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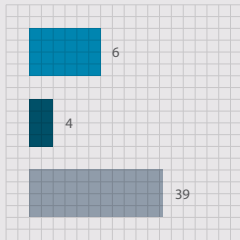
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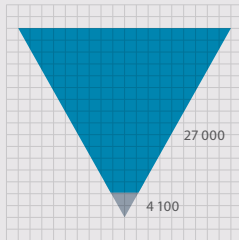
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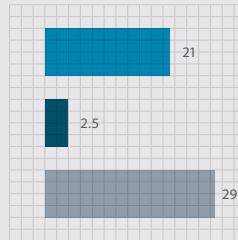
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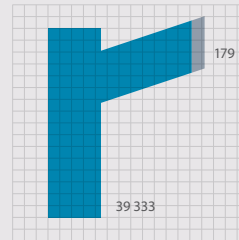
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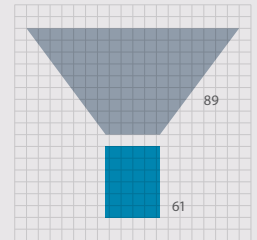
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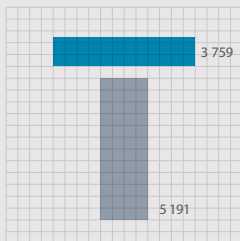
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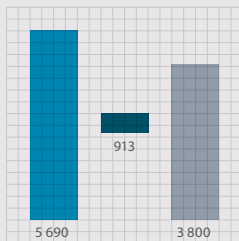
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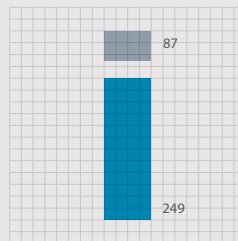
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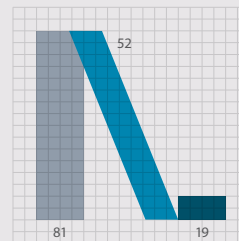
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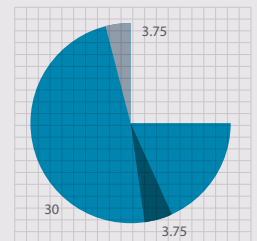
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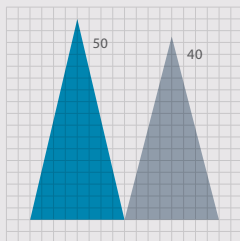
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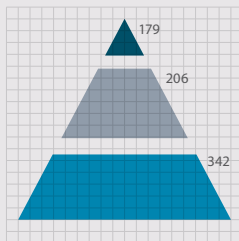
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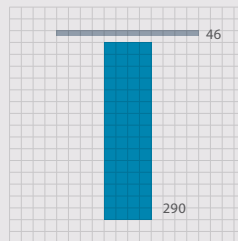
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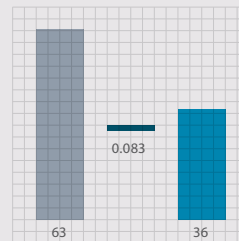
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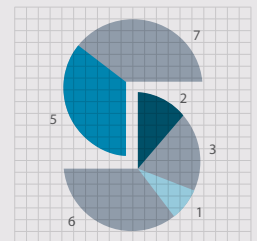
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EVERYTHING MATHS

Mathematics is commonly thought of as being about numbers but mathematics is actually a language! Mathematics is the language that nature speaks to us in. As we learn to understand and speak this language, we can discover many of nature's secrets. Just as understanding someone's language is necessary to learn more about them, mathematics is required to learn about all aspects of the world – whether it is physical sciences, life sciences or even finance and economics.

The great writers and poets of the world have the ability to draw on words and put them together in ways that can tell beautiful or inspiring stories. In a similar way, one can draw on mathematics to explain and create new things. Many of the modern technologies that have enriched our lives are greatly dependent on mathematics. DVDs, Google searches, bank cards with PIN numbers are just some examples. And just as words were not created specifically to tell a story but their existence enabled stories to be told, so the mathematics used to create these technologies was not developed for its own sake, but was available to be drawn on when the time for its application was right.

There is in fact not an area of life that is not affected by mathematics. Many of the most sought after careers depend on the use of mathematics. Civil engineers use mathematics to determine how to best design new structures; economists use mathematics to describe and predict how the economy will react to certain changes; investors use mathematics to price certain types of shares or calculate how risky particular investments are; software developers use mathematics for many of the algorithms (such as Google searches and data security) that make programmes useful.

But, even in our daily lives mathematics is everywhere – in our use of distance, time and money. Mathematics is even present in art, design and music as it informs proportions and musical tones. The greater our ability to understand mathematics, the greater our ability to appreciate beauty and everything in nature. Far from being just a cold and abstract discipline, mathematics embodies logic, symmetry, harmony and technological progress. More than any other language, mathematics is everywhere and universal in its application.

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The merger between Metropolitan and Momentum was lauded for the complementary fit between two companies. This complementary fit is also evident in the focus areas of CSI programmes where Metropolitan and Momentum together cover and support the most important sectors and where the greatest need is in terms of social participation.

HIV/AIDS is becoming a manageable disease in many developed countries but in a country such as ours, it remains a disease where people are still dying of this scourge unnecessarily. Metropolitan continues to make a difference in making sure that HIV AIDS moves away from being a death sentence to a manageable disease. Metropolitan's other focus area is education which remains the key to economic prosperity for our country.

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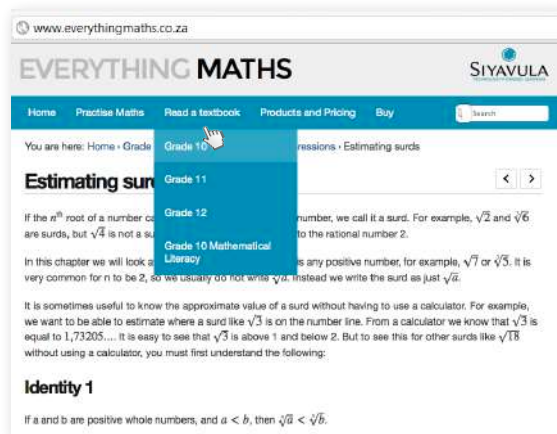
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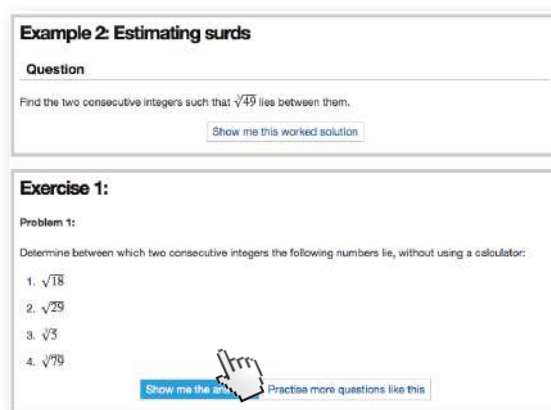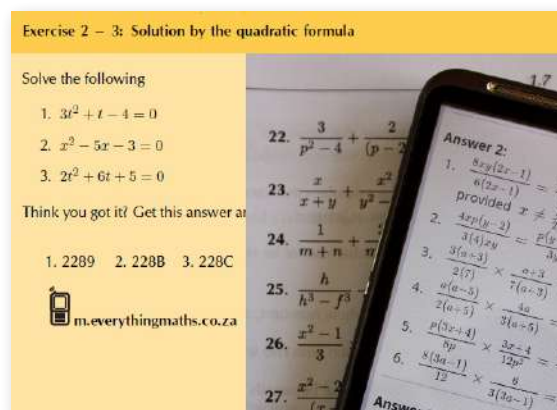
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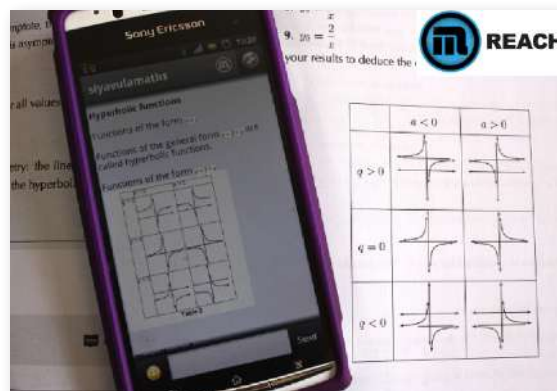
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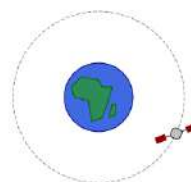
The International Space Station (ISS) has a mass M, as it orbits the Earth, it experiences a gravitational force of F. A space shuttle docks onto the ISS. The gravitational force the ISS experiences once the mass of the shuttle is added increases by a factor of 3.

By what factor does the mass of the ISS increase for it to experience this increase of gravitational force? Write your answer as a fraction of the original mass M_{ISS} of the ISS.

Answer: M_{ISS} [2 points] Check answer

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Contents

CHAPTER

Sequences and series

In earlier grades we learnt about number patterns, which included linear sequences with a common difference and quadratic sequences with a common second difference. We also looked at completing a sequence and how to determine the general term of a sequence.

In this chapter we also look at geometric sequences, which have a constant ratio between consecutive terms. We will learn about arithmetic and geometric series, which are the summing of the terms in sequences.

1.1 Arithmetic sequences EMCDP

An arithmetic sequence is a sequence where consecutive terms are calculated by adding a constant value (positive or negative) to the previous term. We call this constant value the common difference (d).

For example,

$$3; 0; -3; -6; -9; \ldots$$

This is an arithmetic sequence because we add -3 to each term to get the next term:

First term	T_1		3
Second term	T_2	$3 + (-3) =$	0
Third term	T_3	$0 + (-3) =$	-3
Fourth term	T_4	$-3 + (-3) =$	-6
Fifth term	T_5	$-6 + (-3) =$	-9
$\vdots$	$\vdots$	$\vdots$	$\vdots$

See video: 284G at www.everythingmaths.co.za

Exercise 1 – 1: Arithmetic sequences

Find the common difference and write down the next 3 terms of the sequence.

1. $2; 6; 10; 14; 18; 22; \ldots$
2. $-1; -4; -7; -10; -13; -16; \ldots$
3. $-5; -3; -1; 1; 3; \ldots$
4. $-1; 10; 21; 32; 43; 54; \ldots$
5. $a - 3b; a - b; a + b; a + 3b; \ldots$
6. $-2; -\frac{3}{2}; -1; -\frac{1}{2}; 0; \frac{1}{2}; 1; \ldots$
7. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 284H 2. 284J 3. 284K 4. 284M 5. 284N 6. 284P

The general term for an arithmetic sequence EMCDQ

For a general arithmetic sequence with first term a and a common difference d, we can generate the following terms:

$$\begin{aligned}
T_1 &= a \\
T_2 &= T_1 + d = a + d \\
T_3 &= T_2 + d = (a + d) + d = a + 2d \\
T_4 &= T_3 + d = (a + 2d) + d = a + 3d \\
\vdots \quad & \quad \vdots \quad \quad \vdots \quad \quad \vdots \\
T_n &= T_{n-1} + d = (a + (n-2)d) + d = a + (n-1)\,d
\end{aligned}$$

Therefore, the general formula for the n^{th} term of an arithmetic sequence is:

$$T_n = a + (n-1)\,d$$

DEFINITION: *Arithmetic sequence*

An arithmetic (or linear) sequence is an ordered set of numbers (called terms) in which each new term is calculated by adding a constant value to the previous term:

$$T_n = a + (n-1)d$$

where

- T_n is the n^{th} term;
- n is the position of the term in the sequence;
- a is the first term;
- d is the common difference.

Test for an arithmetic sequence

To test whether a sequence is an arithmetic sequence or not, check if the difference between any two consecutive terms is constant:

$$d = T_2 - T_1 = T_3 - T_2 = \ldots = T_n - T_{n-1}$$

If this is not true, then the sequence is not an arithmetic sequence.

Worked example 1: Arithmetic sequence

QUESTION

Given the sequence $-15; -11; -7; \ldots 173$.

1. Is this an arithmetic sequence?
2. Find the formula of the general term.
3. Determine the number of terms in the sequence.

SOLUTION

Step 1: Check if there is a common difference between successive terms

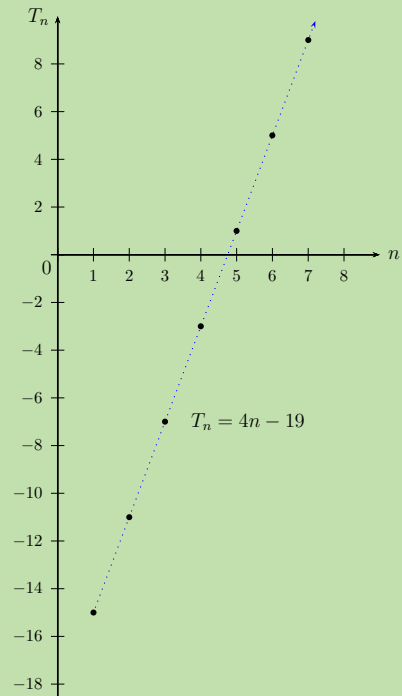
$$T_2 - T_1 = -11 - (-15) = 4$$
$$T_3 - T_2 = -7 - (-11) = 4$$
$$\therefore \text{ This is an arithmetic sequence with } d = 4$$

Step 2: Determine the formula for the general term

Write down the formula and the known values:

$$T_n = a + (n-1)d$$

$$a = -15; \qquad d = 4$$

$$\begin{aligned} T_n &= a + (n-1)d \\ &= -15 + (n-1)(4) \\ &= -15 + 4n - 4 \\ &= 4n - 19 \end{aligned}$$

A graph was not required for this question but it has been included to show that the points of the arithmetic sequence lie in a straight line.

Note: The numbers of the sequence are natural numbers ($n \in \{1; 2; 3; \ldots\}$) and therefore we should not connect the plotted points. In the diagram above, a dotted line has been used to show that the graph of the sequence lies on a straight line.

Step 3: Determine the number of terms in the sequence

$$\begin{aligned} T_n &= a + (n-1)d \\ 173 &= 4n - 19 \\ 192 &= 4n \\ \therefore n &= \frac{192}{4} \\ &= 48 \\ \therefore T_{48} &= 173 \end{aligned}$$

Step 4: Write the final answer

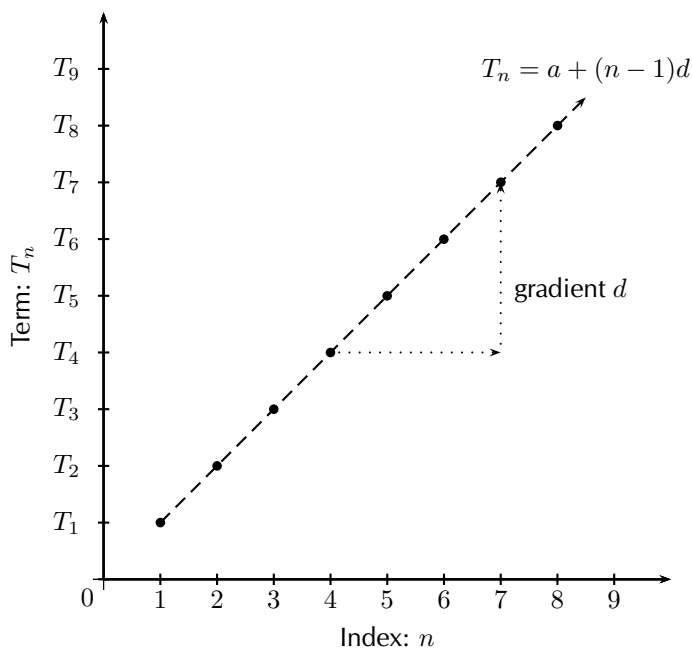
Therefore, there are 48 terms in the sequence.

Arithmetic mean

The arithmetic mean between two numbers is the number half-way between the two numbers. In other words, it is the average of the two numbers. The arithmetic mean and the two terms form an arithmetic sequence.

For example, the arithmetic mean between 7 and 17 is calculated:

$$\begin{aligned}\text{Arithmetic mean} &= \frac{7+17}{2} \\ &= 12 \\ \therefore 7; 12; 17 &\text{ is an arithmetic sequence} \\ T_2 - T_1 &= 12 - 7 = 5 \\ T_3 - T_2 &= 17 - 12 = 5\end{aligned}$$

Plotting a graph of the terms of a sequence sometimes helps in determining the type of sequence involved. For an arithmetic sequence, plotting T_n vs. n results in the following graph:

- If the sequence is arithmetic, the plotted points will lie in a straight line.
- Arithmetic sequences are also called linear sequences, where the common difference (d) is the gradient of the straight line.

$$\begin{aligned}T_n &= a + (n-1)d \\ \text{can be written as } T_n &= d(n-1) + a \\ \text{which is of the same form as } y &= mx + c\end{aligned}$$

Exercise 1 – 2: Arithmetic Sequences

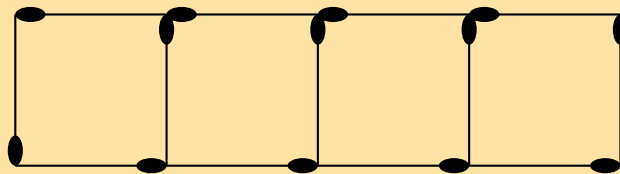
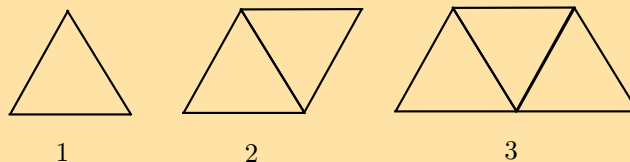
1. Given the sequence $7; 5{,}5; 4; 2{,}5; \ldots$
   a) Find the next term in the sequence.
   b) Determine the general term of the sequence.
   c) Which term has a value of -23?
2. Given the sequence $2; 6; 10; 14; \ldots$
   a) Is this an arithmetic sequence? Justify your answer by calculation.
   b) Calculate T_{55}.
   c) Which term has a value of 322?
   d) Determine by calculation whether or not 1204 is a term in the sequence?
3. An arithmetic sequence has the general term $T_n = -2n + 7$.
   a) Calculate the second, third and tenth terms of the sequence.
   b) Draw a diagram of the sequence for $0 < n \leq 10$.
4. The first term of an arithmetic sequence is $-\frac{1}{2}$ and $T_{22} = 10$. Find T_n.
5. What are the important characteristics of an arithmetic sequence?
6. You are given the first four terms of an arithmetic sequence. Describe the method you would use to find the formula for the n^{th} term of the sequence.
7. A single square is made from 4 matchsticks. To make two squares in a row takes 7 matchsticks, while three squares in a row takes 10 matchsticks.

   a) Write down the first four terms of the sequence.
   b) What is the common difference?
   c) Determine the formula for the general term.
   d) How many matchsticks are in a row of 25 squares?
   e) If there are 109 matchsticks, calculate the number of squares in the row.
8. A pattern of equilateral triangles decorates the border of a girl's skirt. Each triangle is made by three stitches, each having a length of 1 cm.

   a) Complete the table:

Figure no.	1	2	3	q	r	n
No. of stitches	3	5	p	15	71	s

   b) The border of the skirt is 2 m in length. If the entire length of the border is decorated with the triangular pattern, how many stitches will there be?

9. The terms $p; (2p+2); (5p+3)$ form an arithmetic sequence. Find p and the 15^{th} term of the sequence.
   [IEB, Nov 2011]
10. The arithmetic mean of $3a-2$ and x is $4a-4$. Determine the value of x in terms of a.
11. Insert seven arithmetic means between the terms $(3s-t)$ and $(-13s+7t)$.
12. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

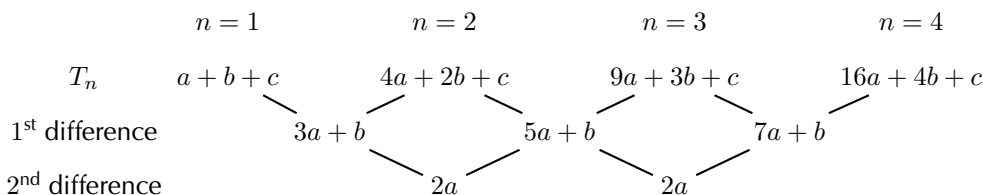
Check answers online with the exercise code below or click on 'show me the answer'.

1. 284Q 2. 284R 3. 284S 4. 284T 5. 284V 6. 284W
7. 284X 8. 284Y 9. 284Z 10. 2852 11. 2853

DEFINITION: *Quadratic sequence*

A quadratic sequence is a sequence of numbers in which the second difference between any two consecutive terms is constant.

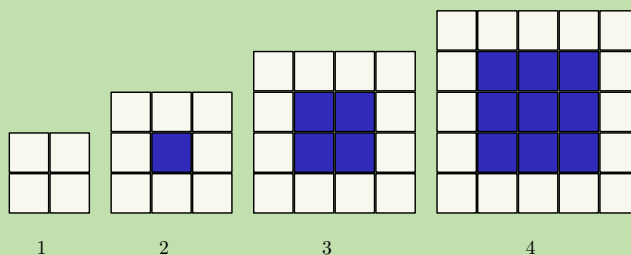
The general formula for the n^{th} term of a quadratic sequence is:

$$T_n = an^2 + bn + c$$

It is important to note that the first differences of a quadratic sequence form an arithmetic sequence. This sequence has a common difference of $2a$ between consecutive terms. In other words, a linear sequence results from taking the first differences of a quadratic sequence.

Worked example 2: Quadratic sequence

QUESTION

Consider the pattern of white and blue blocks in the diagram below.

1. Determine the sequence formed by the white blocks (w).
2. Find the sequence formed by the blue blocks (b).

Pattern number (n)	1	2	3	4	5	6	n
No. of white blocks (w)							
Common difference (d)							

Pattern number (n)	1	2	3	4	5	6	n
No. of blue blocks (b)							
Common difference (d)							

SOLUTION

Step 1: Use the diagram to complete the table for the white blocks

Pattern number (n)	1	2	3	4	5	6	n
No. of white blocks (w)	4	8	12	16	20	24	$4n$
Common difference (d)		4	4	4	4	4	

We see that the next term in the sequence is obtained by adding 4 to the previous term, therefore the sequence is linear and the common difference (d) is 4.

The general term is:

$$\begin{aligned} T_n &= a + (n-1)d \\ &= 4 + (n-1)(4) \\ &= 4 + 4n - 4 \\ &= 4n \end{aligned}$$

Step 2: Use the diagram to complete the table for the blue blocks

Pattern number (n)	1	2	3	4	5	6
No. of blue blocks (b)	0	1	4	9	16	25
Difference		1	3	5	7	9

We notice that there is no common difference between successive terms. However, there is a pattern and on further investigation we see that this is in fact a quadratic sequence:

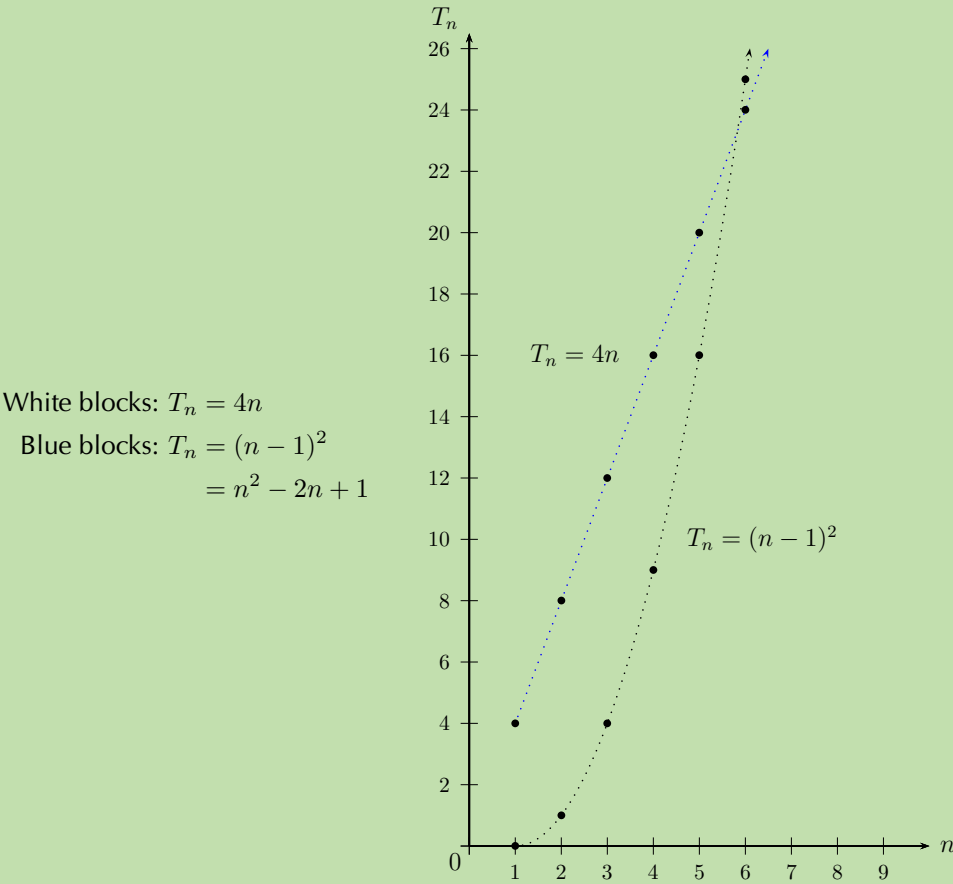
Pattern number (n)	1	2	3	4	5	6	n
No. of blue blocks (b)	0	1	4	9	16	25	$(n-1)^2$
First difference	–	1	3	5	7	9	–
Second difference	–	–	2	2	2	2	–
Pattern	$(1-1)^2$	$(2-1)^2$	$(3-1)^2$	$(4-1)^2$	$(5-1)^2$	$(6-1)^2$	$(n-1)^2$

$$T_n = (n-1)^2$$

Step 3: Draw a graph of T_n vs. n for each sequence

White blocks: $T_n = 4n$

Blue blocks: $T_n = (n-1)^2$

$= n^2 - 2n + 1$

Since the numbers of the sequences are natural numbers ($n \in \{1; 2; 3; \ldots\}$), we should not connect the plotted points. In the diagram above, a dotted line has been used to show that the graph of the sequence formed by the white blocks (w) is a straight line and the graph of the sequence formed by the blue blocks (b) is a parabola.

Exercise 1 – 3: Quadratic sequences

1. Determine whether each of the following sequences is:

   - a linear sequence;
   - a quadratic sequence;
   - or neither.

   a) $8; 17; 32; 53; 80; \ldots$
   b) $3p^2; 6p^2; 9p^2; 12p^2; 15p^2; \ldots$
   c) $1; 2{,}5; 5; 8{,}5; 13; \ldots$
   d) $2; 6; 10; 14; 18; \ldots$
   e) $5; 19; 41; 71; 109; \ldots$
   f) $3; 9; 16; 21; 27; \ldots$
   g) $2k; 8k; 18k; 32k; 50k; \ldots$
   h) $2\frac{1}{2}; 6; 10\frac{1}{2}; 16; 22\frac{1}{2}; \ldots$

2. A quadratic pattern is given by $T_n = n^2 + bn + c$. Find the values of b and c if the sequence starts with the following terms:

$$-1\ ;\ 2\ ;\ 7\ ;\ 14\ ;\ \ldots$$

3. $a^2; -a^2; -3a^2; -5a^2; \ldots$ are the first 4 terms of a sequence.

   a) Is the sequence linear or quadratic? Motivate your answer.
   b) What is the next term in the sequence?
   c) Calculate T_{100}.

4. Given $T_n = n^2 + bn + c$, determine the values of b and c if the sequence starts with the terms:

$$2\ ;\ 7\ ;\ 14\ ;\ 23\ ;\ \ldots$$

5. The first term of a quadratic sequence is 4, the third term is 34 and the common second difference is 10. Determine the first six terms in the sequence.

6. A quadratic sequence has a second term equal to 1, a third term equal to -6 and a fourth term equal to -14.

   a) Determine the second difference for this sequence.
   b) Hence, or otherwise, calculate the first term of the pattern.

7. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 2854 1b. 2855 1c. 2856 1d. 2857 1e. 2858 1f. 2859
1g. 285B 1h. 285C 2. 285D 3. 285F 4. 285G 5. 285H
6. 285J

1.2 Geometric sequences EMCDR

DEFINITION: *Geometric sequence*

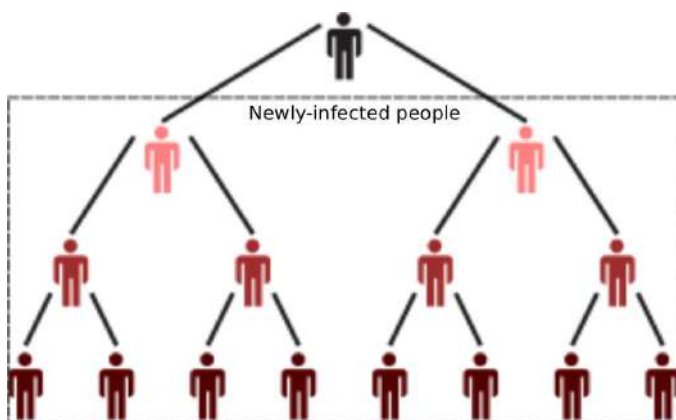
A geometric sequence is a sequence of numbers in which each new term (except for the first term) is calculated by multiplying the previous term by a constant value called the constant ratio (r).

See video: 285K at www.everythingmaths.co.za

This means that the ratio between consecutive numbers in a geometric sequence is a constant (positive or negative). We will explain what we mean by ratio after looking at the following example.

Example: A flu epidemic EMCDS

Influenza (commonly called "flu") is caused by the influenza virus, which infects the respiratory tract (nose, throat, lungs). It can cause mild to severe illness that most of us get during winter time. The influenza virus is spread from person to person in respiratory droplets of coughs and sneezes. This is called "droplet spread". This can happen when droplets from a cough or sneeze of an infected person are propelled through the air and deposited on the mouth or nose of people nearby. It is good practice to cover your mouth when you cough or sneeze so as not to infect others around you when you have the flu. Regular hand washing is an effective way to prevent the spread of infection and illness.

Assume that you have the flu virus, and you forgot to cover your mouth when two friends came to visit while you were sick in bed. They leave, and the next day they also have the flu. Let's assume that each friend in turn spreads the virus to two of their friends by the same droplet spread the following day. Assuming this pattern continues and each sick person infects 2 other friends, we can represent these events in the following manner:

Each person infects two more people with the flu virus.

We can tabulate the events and formulate an equation for the general case:

Day (n)	**No. of newly-infected people**
1	$2 = 2$
2	$4 = 2 \times 2 = 2 \times 2^1$
3	$8 = 2 \times 4 = 2 \times 2 \times 2 = 2 \times 2^2$
4	$16 = 2 \times 8 = 2 \times 2 \times 2 \times 2 = 2 \times 2^3$
5	$32 = 2 \times 16 = 2 \times 2 \times 2 \times 2 \times 2 = 2 \times 2^4$
$\vdots$	$\vdots$
n	$2 \times 2 \times 2 \times 2 \times \cdots \times 2 = 2 \times 2^{n-1}$

The above table represents the number of **newly-infected** people after n days since you first infected your 2 friends.

You sneeze and the virus is carried over to 2 people who start the chain ($a = 2$). The next day, each one then infects 2 of their friends. Now 4 people are newly-infected. Each of them infects 2 people the third day, and 8 new people are infected, and so on. These events can be written as a geometric sequence:

$$2; 4; 8; 16; 32; \ldots$$

Note the constant ratio ($r = 2$) between the events. Recall from the linear arithmetic sequence how the common difference between terms was established. In the geometric sequence we can determine the constant ratio (r) from:

$$\frac{T_2}{T_1} = \frac{T_3}{T_2} = r$$

More generally,

$$\frac{T_n}{T_{n-1}} = r$$

Exercise 1 – 4: Constant ratio of a geometric sequence

Determine the constant ratios for the following geometric sequences and write down the next three terms in each sequence:

1. $5; 10; 20; \ldots$
2. $\frac{1}{2}; \frac{1}{4}; \frac{1}{8}; \ldots$
3. $7; 0{,}7; 0{,}07; \ldots$
4. $p; 3p^2; 9p^3; \ldots$
5. $-3; 30; -300; \ldots$
6. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 285M 2. 285N 3. 285P 4. 285Q 5. 285R

The general term for a geometric sequence EMCDT

From the flu example above we know that $T_1 = 2$ and $r = 2$, and we have seen from the table that the n^{th} term is given by $T_n = 2 \times 2^{n-1}$.

The general geometric sequence can be expressed as:

$$\begin{aligned}
T_1 &= a && = ar^0 \\
T_2 &= a \times r && = ar^1 \\
T_3 &= a \times r \times r && = ar^2 \\
T_4 &= a \times r \times r \times r && = ar^3 \\
T_n &= a \times [r \times r \ldots (n-1) \text{ times}] && = ar^{n-1}
\end{aligned}$$

Therefore the general formula for a geometric sequence is:

$$T_n = ar^{n-1}$$

where

- a is the first term in the sequence;
- r is the constant ratio.

Test for a geometric sequence

To test whether a sequence is a geometric sequence or not, check if the ratio between any two consecutive terms is constant:

$$\frac{T_2}{T_1} = \frac{T_3}{T_2} = \frac{T_n}{T_{n-1}} = r$$

If this condition does not hold, then the sequence is not a geometric sequence.

Exercise 1 – 5: General term of a geometric sequence

Determine the general formula for the n^{th} term of each of the following geometric sequences:

1. $5; 10; 20; \ldots$
2. $\frac{1}{2}; \frac{1}{4}; \frac{1}{8}; \ldots$
3. $7; 0{,}7; 0{,}07; \ldots$
4. $p; 3p^2; 9p^3; \ldots$
5. $-3; 30; -300; \ldots$
6. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 285S 2. 285T 3. 285V 4. 285W 5. 285X

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Worked example 3: Flu epidemic

QUESTION

We continue with the previous flu example, where T_n is the number of newly-infected people after n days:

$$T_n = 2 \times 2^{n-1}$$

1. Calculate how many newly-infected people there are on the tenth day.
2. On which day will $16\ 384$ people be newly-infected?

SOLUTION

Step 1: Write down the known values and the general formula

$$\begin{aligned} a &= 2 \\ r &= 2 \\ T_n &= 2 \times 2^{n-1} \end{aligned}$$

Step 2: Use the general formula to calculate T_{10}

Substitute $n = 10$ into the general formula:

$$\begin{aligned} T_n &= a \times r^{n-1} \\ \therefore T_{10} &= 2 \times 2^{10-1} \\ &= 2 \times 2^9 \\ &= 2 \times 512 \\ &= 1024 \end{aligned}$$

On the tenth day, there are 1024 newly-infected people.

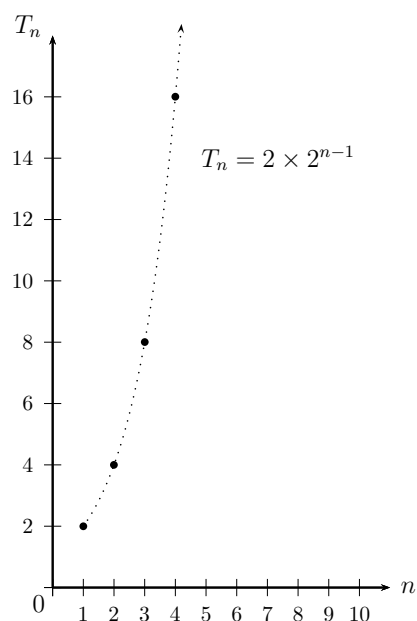
Step 3: Use the general formula to calculate n

We know that $T_n = 16\ 384$ and can use the general formula to calculate the corresponding value of n:

$$\begin{aligned} T_n &= ar^{n-1} \\ 16\ 384 &= 2 \times 2^{n-1} \\ \frac{16\ 384}{2} &= 2^{n-1} \\ 8192 &= 2^{n-1} \\ \text{We can write } 8192 \text{ as } 2^{13} & \\ \text{So } 2^{13} &= 2^{n-1} \\ \therefore 13 &= n - 1 \quad \text{(same bases)} \\ \therefore n &= 14 \end{aligned}$$

There are $16\ 384$ newly-infected people on the 14^{th} day.

For this geometric sequence, plotting the number of newly-infected people (T_n) vs. the number of days (n) results in the following graph:

Day (n)	**No. of newly-infected people**
1	2
2	4
3	8
4	16
5	32
6	64
n	$2 \times 2^{n-1}$

In this example we are only dealing with positive integers ($n \in \{1; 2; 3; \ldots\}, T_n \in \{1; 2; 3; \ldots\}$), therefore the graph is not continuous and we do not join the points with a curve (the dotted line has been drawn to indicate the shape of an exponential graph).

Geometric mean

The geometric mean between two numbers is the value that forms a geometric sequence together with the two numbers.

For example, the geometric mean between 5 and 20 is the number that has to be inserted between 5 and 20 to form the geometric sequence: $5; x; 20$

$$\text{Determine the constant ratio: } \frac{x}{5} = \frac{20}{x}$$
$$\therefore x^2 = 20 \times 5$$
$$x^2 = 100$$
$$x = \pm 10$$

Important: remember to include both the positive and negative square root. The geometric mean generates two possible geometric sequences:

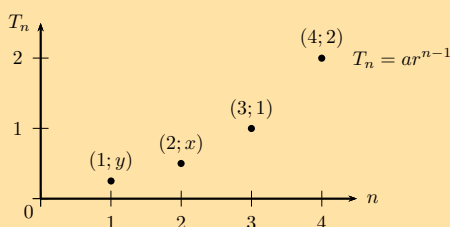
$$5; 10; 20; \ldots$$
$$5; -10; 20; \ldots$$

In general, the geometric mean (x) between two numbers a and b forms a geometric sequence with a and b:

$$\text{For a geometric sequence: } a; x; b$$

$$\text{Determine the constant ratio: } \frac{x}{a} = \frac{b}{x}$$
$$x^2 = ab$$
$$\therefore x = \pm\sqrt{ab}$$

Exercise 1 – 6: Mixed exercises

1. The n^{th} term of a sequence is given by the formula $T_n = 6\left(\frac{1}{3}\right)^{n-1}$.
   a) Write down the first three terms of the sequence.
   b) What type of sequence is this?
2. Consider the following terms:
   $$(k-4); (k+1); m; 5k$$
   The first three terms form an arithmetic sequence and the last three terms form a geometric sequence. Determine the values of k and m if both are positive integers.
   [IEB, Nov 2006]
3. Given a geometric sequence with second term $\frac{1}{2}$ and ninth term 64.
   a) Determine the value of r.
   b) Find the value of a.
   c) Determine the general formula of the sequence.
4. The diagram shows four sets of values of consecutive terms of a geometric sequence with the general formula $T_n = ar^{n-1}$.

   a) Determine a and r.
   b) Find x and y.
   c) Find the fifth term of the sequence.
5. Write down the next two terms for the following sequence:
   $$1; \sin\theta; 1-\cos^2\theta; \ldots$$
6. $5; x; y$ is an arithmetic sequence and $x; y; 81$ is a geometric sequence. All terms in the sequences are integers. Calculate the values of x and y.
7. The two numbers $2x^2y^2$ and $8x^4$ are given.
   a) Write down the geometric mean between the two numbers in terms of x and y.
   b) Determine the constant ratio of the resulting sequence.
8. Insert three geometric means between -1 and $-\frac{1}{81}$. Give all possible answers.
9. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.
1. 285Y 2. 285Z 3. 2862 4. 2863 5. 2864 6. 2865
7. 2866 8. 2867

1.3 Series EMCDV

It is often important and valuable to determine the sum of the terms of an arithmetic or geometric sequence. The sum of any sequence of numbers is called a series.

Finite series

We use the symbol S_n for the sum of the first n terms of a sequence $\{T_1; T_2; T_3; \ldots; T_n\}$:

$$S_n = T_1 + T_2 + T_3 + \cdots + T_n$$

If we sum only a finite number of terms, we get a finite series.

For example, consider the following sequence of numbers

$$1; 4; 9; 16; 25; 36; 49; \ldots$$

We can calculate the sum of the first four terms:

$$S_4 = 1 + 4 + 9 + 16 = 30$$

This is an example of a finite series since we are only summing four terms.

Infinite series

If we sum infinitely many terms of a sequence, we get an infinite series:

$$S_\infty = T_1 + T_2 + T_3 + \cdots$$

Sigma notation EMCDW

Sigma notation is a very useful and compact notation for writing the sum of a given number of terms of a sequence.

A sum may be written out using the summation symbol $\sum$ (Sigma), which is the capital letter "S" in the Greek alphabet. It indicates that you must sum the expression to the right of the summation symbol:

For example,

$$\sum_{n=1}^{5} 2n = 2 + 4 + 6 + 8 + 10 = 30$$

In general,

$$\sum_{i=m}^{n} T_i = T_m + T_{m+1} + \cdots + T_{n-1} + T_n$$

where

- i is the index of the sum;
- m is the lower bound (or start index), shown below the summation symbol;
- n is the upper bound (or end index), shown above the summation symbol;
- T_i is a term of a sequence;
- the number of terms in the series $=$ end index $-$ start index $+ 1$.

The index i increases from m to n by steps of 1.

Note that this is also sometimes written as:

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \cdots + a_{n-1} + a_n$$

When we write out all the terms in a sum, it is referred to as the expanded form.

If we are summing from $i = 1$ (which implies summing from the first term in a sequence), then we can use either S_n or $\sum$ notation:

$$S_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n \quad (n \text{ terms})$$

Worked example 4: Sigma notation

QUESTION

Expand the sequence and find the value of the series:

$$\sum_{n=1}^{6} 2^n$$

SOLUTION

Step 1: Expand the formula and write down the first six terms of the sequence

$$\begin{aligned}\sum_{n=1}^{6} 2^n &= 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 \quad (6 \text{ terms}) \\ &= 2 + 4 + 8 + 16 + 32 + 64\end{aligned}$$

This is a geometric sequence $2; 4; 8; 16; 32; 64$ with a constant ratio of 2 between consecutive terms.

Step 2: Determine the sum of the first six terms of the sequence

$$\begin{aligned}S_6 &= 2 + 4 + 8 + 16 + 32 + 64 \\ &= 126\end{aligned}$$

Worked example 5: Sigma notation

QUESTION

Find the value of the series:

$$\sum_{n=3}^{7} 2an$$

SOLUTION

Step 1: Expand the sequence and write down the five terms

$$\begin{aligned}\sum_{n=3}^{7} 2an &= 2a(3) + 2a(4) + 2a(5) + 2a(6) + 2a(7) \quad \text{(5 terms)} \\ &= 6a + 8a + 10a + 12a + 14a\end{aligned}$$

Step 2: Determine the sum of the five terms of the sequence

$$\begin{aligned}S_5 &= 6a + 8a + 10a + 12a + 14a \\ &= 50a\end{aligned}$$

Worked example 6: Sigma notation

QUESTION

Write the following series in sigma notation:

$$31 + 24 + 17 + 10 + 3$$

SOLUTION

Step 1: Consider the series and determine if it is an arithmetic or geometric series

First test for an arithmetic series: is there a common difference?

We let:

$$\begin{aligned}&T_1 = 31; \quad T_4 = 10; \\ &T_2 = 24; \quad T_5 = 3; \\ &T_3 = 17;\end{aligned}$$

We calculate:

$$\begin{aligned}d &= T_2 - T_1 \\ &= 24 - 31 \\ &= -7 \\ d &= T_3 - T_2 \\ &= 17 - 24 \\ &= -7\end{aligned}$$

There is a common difference of -7, therefore this is an arithmetic series.

Step 2: Determine the general formula of the series

$$\begin{aligned}T_n &= a + (n-1)d \\ &= 31 + (n-1)(-7) \\ &= 31 - 7n + 7 \\ &= -7n + 38\end{aligned}$$

Be careful: brackets must be used when substituting $d = -7$ into the general term. Otherwise the equation would be $T_n = 31 + (n-1) - 7$, which would be incorrect.

Step 3: Determine the sum of the series and write in sigma notation

$$31 + 24 + 17 + 10 + 3 = 85$$

$$\therefore \sum_{n=1}^{5} (-7n + 38) = 85$$

Rules for sigma notation

1. Given two sequences, a_i and b_i:

$$\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

2. For any constant c that is not dependent on the index i:

$$\begin{aligned}\sum_{i=1}^{n}(c \,.\, a_i) &= c \,.\, a_1 + c \,.\, a_2 + c \,.\, a_3 + \cdots + c \,.\, a_n \\ &= c\,(a_1 + a_2 + a_3 + \cdots + a_n) \\ &= c\sum_{i=1}^{n} a_i\end{aligned}$$

3. Be accurate with the use of brackets:

   Example 1:

$$\begin{aligned}\sum_{n=1}^{3} (2n+1) &= 3 + 5 + 7 \\ &= 15\end{aligned}$$

   Example 2:

$$\begin{aligned}\sum_{n=1}^{3} (2n) + 1 &= (2 + 4 + 6) + 1 \\ &= 13\end{aligned}$$

   Note: the series in the second example has the general term $T_n = 2n$ and the $+1$ is added to the sum of the three terms. It is very important in sigma notation to use brackets correctly.

4. $\sum_{i=m}^{n} a_i$

The values of i:

- start at m (m is not always 1);
- increase in steps of 1;
- and end at n.

Exercise 1 – 7: Sigma notation

1. Determine the value of the following:

   a) $$\sum_{k=1}^{4} 2$$

   b) $$\sum_{i=-1}^{3} i$$

   c) $$\sum_{n=2}^{5} (3n-2)$$

2. Expand the series:

   a) $$\sum_{k=1}^{6} 0^k$$

   b) $$\sum_{n=-3}^{0} 8$$

   c) $$\sum_{k=1}^{5} (ak)$$

3. Calculate the value of a:

   a) $$\sum_{k=1}^{3} \left(a \,.\, 2^{k-1}\right) = 28$$

   b) $$\sum_{j=1}^{4} \left(2^{-j}\right) = a$$

4. Write the following in sigma notation:
   $$\frac{1}{9} + \frac{1}{3} + 1 + 3$$

5. Write the sum of the first 25 terms of the series below in sigma notation:
   $$11 + 4 - 3 - 10 \ldots$$

6. Write the sum of the first 1000 natural, odd numbers in sigma notation.

7. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 2868 1b. 2869 1c. 286B 2a. 286C 2b. 286D 2c. 286F
3a. 286G 3b. 286H 4. 286J 5. 286K 6. 286M

1.4 Finite arithmetic series EMCDX

An arithmetic sequence is a sequence of numbers, such that the difference between any term and the previous term is a constant number called the common difference (d):

$$T_n = a + (n-1)\,d$$

where

- T_n is the n^{th} term of the sequence;
- a is the first term;
- d is the common difference.

When we sum a finite number of terms in an arithmetic sequence, we get a finite arithmetic series.

The sum of the first one hundred integers

A simple arithmetic sequence is when $a = 1$ and $d = 1$, which is the sequence of positive integers:

$$\begin{aligned} T_n &= a + (n-1)\,d \\ &= 1 + (n-1)\,(1) \\ &= n \\ \therefore \{T_n\} &= 1; 2; 3; 4; 5; \ldots \end{aligned}$$

If we wish to sum this sequence from $n = 1$ to any positive integer, for example 100, we would write

$$\sum_{n=1}^{100} n = 1 + 2 + 3 + \cdots + 100$$

This gives the answer to the sum of the first 100 positive integers.

The mathematician, Karl Friedrich Gauss, discovered the following proof when he was only 8 years old. His teacher had decided to give his class a problem which would distract them for the entire day by asking them to add all the numbers from 1 to 100. Young Karl quickly realised how to do this and shocked the teacher with the correct answer, 5050. This is the method that he used:

- Write the numbers in ascending order.
- Write the numbers in descending order.
- Add the corresponding pairs of terms together.
- Simplify the equation by making S_n the subject of the equation.

$$
\begin{aligned}
S_{100} &= 1 \quad + \quad 2 \quad + \quad 3 + \cdots + 98 + 99 + 100 \\
+ \quad \underline{S_{100}} &= \underline{100 + 99 + 98 + \cdots + 3 \quad + \quad 2 \quad + \quad 1} \\
\therefore 2S_{100} &= 101 + 101 + 101 + \cdots + 101 + 101 + 101 \\
\therefore 2S_{100} &= 101 \times 100 \\
&= 10\ 100 \\
\therefore S_{100} &= \frac{10\ 100}{2} \\
&= 5050
\end{aligned}
$$

General formula for a finite arithmetic series EMCDY

If we sum an arithmetic sequence, it takes a long time to work it out term-by-term. We therefore derive the general formula for evaluating a finite arithmetic series. We start with the general formula for an arithmetic sequence of n terms and sum it from the first term (a) to the last term in the sequence (l):

$$
\begin{aligned}
\sum_{n=1}^{l} T_n &= S_n \\
S_n &= a + (a+d) + (a+2d) + \cdots + (l-2d) + (l-d) + l \\
+ \quad \underline{S_n} &= \underline{l + (l-d) + (l-2d) + \cdots + (a+2d) + (a+d) + a} \\
\therefore 2S_n &= (a+l) + (a+l) + (a+l) + \cdots + (a+l) + (a+l) + (a+l) \\
\therefore 2S_n &= n \times (a+l) \\
\therefore S_n &= \frac{n}{2}(a+l)
\end{aligned}
$$

This general formula is useful if the last term in the series is known.

We substitute $l = a + (n-1)d$ into the above formula and simplify:

$$
\begin{aligned}
S_n &= \frac{n}{2}(a + [a + (n-1)d]) \\
\therefore S_n &= \frac{n}{2}[2a + (n-1)d]
\end{aligned}
$$

The general formula for determining the sum of an arithmetic series is given by:

$$S_n = \frac{n}{2}\left[2a + (n-1)\,d\right]$$

or

$$S_n = \frac{n}{2}(a+l)$$

For example, we can calculate the sum S_{20} for the arithmetic sequence $T_n = 3 + 7\,(n-1)$ by summing all the individual terms:

$$
\begin{aligned}
S_{20} &= \sum_{n=1}^{20} \left[3 + 7\,(n-1)\right] \\
&= 3 + 10 + 17 + 24 + 31 + 38 + 45 + 52 \\
&+ 59 + 66 + 73 + 80 + 87 + 94 + 101 \\
&+ 108 + 115 + 122 + 129 + 136 \\
&= 1390
\end{aligned}
$$

or, more sensibly, we could use the general formula for determining an arithmetic series by substituting $a = 3$, $d = 7$ and $n = 20$:

$$\begin{aligned} S_n &= \frac{n}{2}(2a + (n-1)d) \\ S_{20} &= \frac{20}{2}\left[2(3) + 7\left(20 - 1\right)\right] \\ &= 1390 \end{aligned}$$

This example demonstrates how useful the general formula for determining an arithmetic series is, especially when the series has a large number of terms.

See video: 286N at www.everythingmaths.co.za

Worked example 7: General formula for the sum of an arithmetic sequence

QUESTION

Find the sum of the first 30 terms of an arithmetic series with $T_n = 7n - 5$ by using the formula.

SOLUTION

Step 1: Use the general formula to generate terms of the sequence and write down the known variables

$$\begin{aligned} T_n &= 7n - 5 \\ \therefore T_1 &= 7(1) - 5 \\ &= 2 \\ T_2 &= 7(2) - 5 \\ &= 9 \\ T_3 &= 7(3) - 5 \\ &= 16 \end{aligned}$$

This gives the sequence: $2; 9; 16 \ldots$

$$a = 2; \quad d = 7; \quad n = 30$$

Step 2: Write down the general formula and substitute the known values

$$\begin{aligned} S_n &= \frac{n}{2}[2a + (n-1)d] \\ S_{30} &= \frac{30}{2}[2(2) + (30-1)(7)] \\ &= 15(4 + 203) \\ &= 15(207) \\ &= 3105 \end{aligned}$$

Step 3: Write the final answer

$S_{30} = 3105$

Worked example 8: Sum of an arithmetic sequence if first and last terms are known

QUESTION

Find the sum of the series $-5-3-1+\cdots\cdots+123$

SOLUTION

Step 1: Identify the type of series and write down the known variables

$$\begin{aligned} d &= T_2 - T_1 \\ &= -3-(-5) \\ &= 2 \\ d &= T_3 - T_2 \\ &= -1-(-3) \\ &= 2 \end{aligned}$$

$$a=-5; \quad d=2; \quad l=123$$

Step 2: Determine the value of n

$$\begin{aligned} T_n &= a+(n-1)d \\ \therefore 123 &= -5+(n-1)(2) \\ &= -5+2n-2 \\ \therefore 130 &= 2n \\ \therefore n &= 65 \end{aligned}$$

Step 3: Use the general formula to find the sum of the series

$$\begin{aligned} S_n &= \frac{n}{2}(a+l) \\ S_{65} &= \frac{65}{2}(-5+123) \\ &= \frac{65}{2}(118) \\ &= 3835 \end{aligned}$$

Step 4: Write the final answer

$S_{65} = 3835$

Worked example 9: Finding n given the sum of an arithmetic sequence

QUESTION

Given an arithmetic sequence with $T_2 = 7$ and $d = 3$, determine how many terms must be added together to give a sum of 2146.

SOLUTION

Step 1: Write down the known variables

$$\begin{aligned} d &= T_2 - T_1 \\ \therefore 3 &= 7 - a \\ \therefore a &= 4 \end{aligned}$$

$$a=4; \quad d=3; \quad S_n=2146$$

Step 2: Use the general formula to determine the value of n

$$\begin{aligned} S_n &= \frac{n}{2}(2a + (n-1)d) \\ 2146 &= \frac{n}{2}(2(4) + (n-1)(3)) \\ 4292 &= n(8 + 3n - 3) \\ \therefore 0 &= 3n^2 + 5n - 4292 \\ &= (3n + 116)(n - 37) \\ \therefore n = -\frac{116}{3} &\text{ or } n = 37 \end{aligned}$$

but n must be a positive integer, therefore $n = 37$.

We could have solved for n using the quadratic formula but factorising by inspection is usually the quickest method.

Step 3: Write the final answer

$S_{37} = 2146$

Worked example 10: Finding n given the sum of an arithmetic sequence

QUESTION

The sum of the second and third terms of an arithmetic sequence is equal to zero and the sum of the first 36 terms of the series is equal to 1152. Find the first three terms in the series.

SOLUTION

Step 1: Write down the given information

$$\begin{aligned} T_2 + T_3 &= 0 \\ \text{So} \quad (a + d) + (a + 2d) &= 0 \\ \therefore 2a + 3d &= 0 \ldots\ldots (1) \end{aligned}$$

$$\begin{aligned} S_n &= \frac{n}{2}(2a + (n-1)d) \\ S_{36} &= \frac{36}{2}(2a + (36-1)d) \\ 1152 &= 18(2a + 35d) \\ \therefore 64 &= 2a + 35d \ldots\ldots (2) \end{aligned}$$

Step 2: Solve the two equations simultaneously

$$\begin{aligned} 2a + 3d &= 0 \ldots\ldots (1) \\ 2a + 35d &= 64 \ldots\ldots (2) \\ \text{Eqn } (2) - (1): \quad 32d &= 64 \\ \therefore d &= 2 \\ \text{And } 2a + 3(2) &= 0 \\ 2a &= -6 \\ \therefore a &= -3 \end{aligned}$$

Step 3: Write the final answer

The first three terms of the series are:

$$T_1 = a = -3$$
$$T_2 = a + d = -3 + 2 = -1$$
$$T_3 = a + 2d = -3 + 2(2) = 1$$
$$-3 - 1 + 1$$

Calculating the value of a term given the sum of n terms:

If the first term in a series is T_1, then $S_1 = T_1$.

We also know the sum of the first two terms $S_2 = T_1 + T_2$, which we rearrange to make T_2 the subject of the equation:

$$T_2 = S_2 - T_1$$
$$\text{Substitute } S_1 = T_1$$
$$\therefore T_2 = S_2 - S_1$$

Similarly, we could determine the third and fourth term in a series:

$$T_3 = S_3 - S_2$$
$$\text{And } T_4 = S_4 - S_3$$

$T_n = S_n - S_{n-1}$, for $n \in \{2; 3; 4; \ldots\}$ and $T_1 = S_1$

Exercise 1 – 8: Sum of an arithmetic series

1. Determine the value of k:
$$\sum_{n=1}^{k} (-2n) = -20$$
2. The sum to n terms of an arithmetic series is $S_n = \frac{n}{2}(7n + 15)$.
    a) How many terms of the series must be added to give a sum of 425?
    b) Determine the sixth term of the series.
3. a) The common difference of an arithmetic series is 3. Calculate the values of n for which the n^{th} term of the series is 93, and the sum of the first n terms is 975.
    b) Explain why there are two possible answers.
4. The third term of an arithmetic sequence is -7 and the seventh term is 9. Determine the sum of the first 51 terms of the sequence.
5. Calculate the sum of the arithmetic series $4 + 7 + 10 + \cdots + 901$.
6. Evaluate without using a calculator: $\dfrac{4 + 8 + 12 + \cdots + 100}{3 + 10 + 17 + \cdots + 101}$

7. The second term of an arithmetic sequence is -4 and the sum of the first six terms of the series is 21.
   a) Find the first term and the common difference.
   b) Hence determine T_{100}.
      [IEB, Nov 2004]
8. Determine the value of the following:
   a) $$\sum_{w=0}^{8} (7w + 8)$$
   b) $$\sum_{j=1}^{8} 7j + 8$$
9. Determine the value of n.
   $$\sum_{c=1}^{n} (2 - 3c) = -330$$
10. The sum of n terms of an arithmetic series is $5n^2 - 11n$ for all values of n. Determine the common difference.
11. The sum of an arithmetic series is 100 times its first term, while the last term is 9 times the first term. Calculate the number of terms in the series if the first term is not equal to zero.
12. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 286P 2a. 286Q 2b. 286R 3. 286S 4. 286T 5. 286V
6. 286W 7. 286X 8a. 286Y 8b. 286Z 9. 2872 10. 2873
11. 2874

1.5 Finite geometric series EMCDZ

When we sum a known number of terms in a geometric sequence, we get a finite geometric series. We generate a geometric sequence using the general form:

$$T_n = a \cdot r^{n-1}$$

where

- n is the position of the sequence;
- T_n is the n^{th} term of the sequence;
- a is the first term;
- r is the constant ratio.

General formula for a finite geometric series EMCF2

$$
\begin{aligned}
S_n &= a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1} \ldots (1) \\
r \times S_n &= \qquad ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1} + ar^n \ldots\ldots (2)
\end{aligned}
$$

Subtract eqn. (2) from eqn. (1)

$$
\begin{aligned}
\therefore S_n - rS_n &= a + 0 + 0 + \cdots - ar^n \\
S_n - rS_n &= a - ar^n \\
S_n(1-r) &= a(1-r^n) \\
\therefore S_n &= \frac{a(1-r^n)}{1-r} \quad \text{(where } r \neq 1\text{)}
\end{aligned}
$$

The general formula for determining the sum of a geometric series is given by:

$$
S_n = \frac{a(1-r^n)}{1-r} \qquad \text{where } r \neq 1
$$

This formula is easier to use when $r < 1$.

Alternative formula:

$$
\begin{aligned}
S_n &= a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1} \ldots\ldots (1) \\
r \times S_n &= \qquad ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1} + ar^n \ldots\ldots (2)
\end{aligned}
$$

Subtract eqn. (1) from eqn. (2)

$$
\begin{aligned}
\therefore rS_n - S_n &= ar^n - a \\
S_n(r-1) &= a(r^n-1) \\
\therefore S_n &= \frac{a(r^n-1)}{r-1} \quad \text{(where } r \neq 1\text{)}
\end{aligned}
$$

The general formula for determining the sum of a geometric series is given by:

$$
S_n = \frac{a(r^n-1)}{r-1} \qquad \text{where } r \neq 1
$$

This formula is easier to use when $r > 1$.

Worked example 11: Sum of a geometric series

QUESTION

Calculate:

$$\sum_{k=1}^{6} 32\left(\frac{1}{2}\right)^{k-1}$$

SOLUTION

Step 1: Write down the first three terms of the series

$$\begin{aligned}
k = 1; \quad T_1 &= 32\left(\frac{1}{2}\right)^{0} = 32\\
k = 2; \quad T_2 &= 32\left(\frac{1}{2}\right)^{2-1} = 16\\
k = 3; \quad T_3 &= 32\left(\frac{1}{2}\right)^{3-1} = 8
\end{aligned}$$

We have generated the series $32 + 16 + 8 + \cdots$

Step 2: Determine the values of a and r

$$\begin{aligned}
a &= T_1 = 32\\
r &= \frac{T_2}{T_1} = \frac{T_3}{T_2} = \frac{1}{2}
\end{aligned}$$

Step 3: Use the general formula to find the sum of the series

$$\begin{aligned}
S_n &= \frac{a(1 - r^n)}{1 - r}\\
S_6 &= \frac{32(1 - \left(\frac{1}{2}\right)^6)}{1 - \frac{1}{2}}\\
&= \frac{32\left(1 - \frac{1}{64}\right)}{\frac{1}{2}}\\
&= 2 \times 32\left(\frac{63}{64}\right)\\
&= 64\left(\frac{63}{64}\right)\\
&= 63
\end{aligned}$$

Step 4: Write the final answer

$$\sum_{k=1}^{6} 32\left(\frac{1}{2}\right)^{k-1} = 63$$

See video: 2875 at www.everythingmaths.co.za

Worked example 12: Sum of a geometric series

QUESTION

Given a geometric series with $T_1 = -4$ and $T_4 = 32$. Determine the values of r and n if $S_n = 84$.

SOLUTION

Step 1: Determine the values of a and r

$$\begin{aligned} a &= T_1 = -4 \\ T_4 &= ar^3 = 32 \\ \therefore -4r^3 &= 32 \\ r^3 &= -8 \\ \therefore r &= -2 \end{aligned}$$

Therefore the geometric series is $-4 + 8 - 16 + 32 \ldots$ Notice that the signs of the terms alternate because $r < 0$.

We write the general term for this series as $T_n = -4(-2)^{n-1}$.

Step 2: Use the general formula for the sum of a geometric series to determine the value of n

$$\begin{aligned} S_n &= \frac{a(1 - r^n)}{1 - r} \\ \therefore 84 &= \frac{-4(1 - (-2)^n)}{1 - (-2)} \\ 84 &= \frac{-4(1 - (-2)^n)}{3} \\ -\frac{3}{4} \times 84 &= 1 - (-2)^n \\ -63 &= 1 - (-2)^n \\ (-2)^n &= 64 \\ (-2)^n &= (-2)^6 \\ \therefore n &= 6 \end{aligned}$$

Step 3: Write the final answer

$$r = -2 \text{ and } n = 6$$

Worked example 13: Sum of a geometric series

QUESTION

Use the general formula for the sum of a geometric series to determine k if

$$\sum_{n=1}^{8} k\left(\frac{1}{2}\right)^{n} = \frac{255}{64}$$

SOLUTION

Step 1: Write down the first three terms of the series

$$\begin{aligned} n = 1; \quad T_1 &= k\left(\frac{1}{2}\right)^{1} = \frac{1}{2}k \\ n = 2; \quad T_2 &= k\left(\frac{1}{2}\right)^{2} = \frac{1}{4}k \\ n = 3; \quad T_3 &= k\left(\frac{1}{2}\right)^{3} = \frac{1}{8}k \end{aligned}$$

We have generated the series $\frac{1}{2}k + \frac{1}{4}k + \frac{1}{8}k + \cdots$

We can take out the common factor k and write the series as: $k\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\right)$

$$\therefore k\sum_{n=1}^{8}\left(\frac{1}{2}\right)^{n} = \frac{255}{64}$$

Step 2: Determine the values of a and r

$$\begin{aligned} a &= T_1 = \frac{1}{2} \\ r &= \frac{T_2}{T_1} = \frac{T_3}{T_2} = \frac{1}{2} \end{aligned}$$

Step 3: Calculate the sum of the first eight terms of the geometric series

$$\begin{aligned} \therefore S_n &= \frac{a(1 - r^n)}{1 - r} \\ S_8 &= \frac{\frac{1}{2}(1 - \left(\frac{1}{2}\right)^8)}{1 - \frac{1}{2}} \\ &= \frac{\frac{1}{2}(1 - \left(\frac{1}{2}\right)^8)}{\frac{1}{2}} \\ &= 1 - \frac{1}{256} \\ &= \frac{255}{256} \end{aligned}$$

$$\therefore \sum_{n=1}^{8}\left(\frac{1}{2}\right)^{n} = \frac{255}{256}$$

So then we can write:

$$k\sum_{n=1}^{8}\left(\frac{1}{2}\right)^{n} = \frac{255}{64}$$
$$k\left(\frac{255}{256}\right) = \frac{255}{64}$$
$$\therefore k = \frac{255}{64} \times \frac{256}{255}$$
$$= \frac{256}{64}$$
$$= 4$$

Step 4: Write the final answer

$$k = 4$$

Exercise 1 – 9: Sum of a geometric series

1. Prove that $a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(r^n-1)}{r-1}$ and state any restrictions.
2. Given the geometric sequence $1; -3; 9; \ldots$ determine:
   a) The eighth term of the sequence.
   b) The sum of the first eight terms of the sequence.
3. Determine:
$$\sum_{n=1}^{4} 3 \,.\, 2^{n-1}$$
4. Find the sum of the first 11 terms of the geometric series $6 + 3 + \frac{3}{2} + \frac{3}{4} + \cdots$
5. Show that the sum of the first n terms of the geometric series $54 + 18 + 6 + \cdots + 5\left(\frac{1}{3}\right)^{n-1}$ is given by $\left(81 - 3^{4-n}\right)$.
6. The eighth term of a geometric sequence is 640. The third term is 20. Find the sum of the first 7 terms.
7. Given:
$$\sum_{t=1}^{n} 8\left(\frac{1}{2}\right)^{t}$$
   a) Find the first three terms in the series.
   b) Calculate the number of terms in the series if $S_n = 7\frac{63}{64}$.
8. The ratio between the sum of the first three terms of a geometric series and the sum of the 4^{th}, 5^{th} and 6^{th} terms of the same series is $8 : 27$. Determine the constant ratio and the first 2 terms if the third term is 8.
9. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 2876 2a. 2877 2b. 2878 3. 2879 4. 287B 5. 287C
6. 287D 7a. 287F 7b. 287G 8. 287H

1.6 Infinite series EMCF3

So far we have been working only with finite sums, meaning that whenever we determined the sum of a series, we only considered the sum of the first n terms. We now consider what happens when we add an infinite number of terms together. Surely if we sum infinitely many numbers, no matter how small they are, the answer goes to infinity? In some cases the answer does indeed go to infinity (like when we sum all the positive integers), but surprisingly there are some cases where the answer is a finite real number.

Investigation: Sum of an infinite series

1. Cut a piece of string 1 m in length.
2. Now cut the piece of string in half and place one half on the desk.
3. Cut the other half in half again and put one of the pieces on the desk.
4. Repeat this process until the piece of string is too short to cut easily.
5. Draw a diagram to illustrate the sequence of lengths of the pieces of string.
6. Can this sequence be expressed mathematically? Hint: express the shorter lengths of string as a fraction of the original length of string.
7. What is the sum of the lengths of all the pieces of string?
8. Predict what would happen if these steps could be repeated infinitely many times.
9. Will the sum of the lengths of string ever be greater than 1?
10. What can you conclude?

Worked example 14: Sum to infinity

QUESTION

Complete the table below for the geometric series $T_n = \left(\frac{1}{2}\right)^n$ and answer the questions that follow:

	Terms	S_n	$1 - S_n$
T_1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$T_1 + T_2$			
$T_1 + T_2 + T_3$			
$T_1 + T_2 + T_3 + T_4$			

1. As more and more terms are added, what happens to the value of S_n?
2. As more more and more terms are added, what happens to the value of $1 - S_n$?
3. Predict the maximum value of S_n for the sum of infinitely many terms in the series.

SOLUTION

Step 1: Complete the table

	Terms	S_n	$1 - S_n$
T_1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$T_1 + T_2$	$\frac{1}{2} + \frac{1}{4}$	$\frac{3}{4}$	$\frac{1}{4}$
$T_1 + T_2 + T_3$	$\frac{1}{2} + \frac{1}{4} + \frac{1}{8}$	$\frac{7}{8}$	$\frac{1}{8}$
$T_1 + T_2 + T_3 + T_4$	$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$	$\frac{15}{16}$	$\frac{1}{16}$

Step 2: Consider the value of S_n and $1 - S_n$

As more terms in the series are added together, the value of S_n increases:

$$\frac{1}{2} \quad < \quad \frac{3}{4} \quad < \quad \frac{7}{8} \quad < \quad \cdots$$

However, by considering $1 - S_n$, we notice that the amount by which S_n increases gets smaller and smaller as more terms are added:

$$\frac{1}{2} \quad > \quad \frac{1}{4} \quad > \quad \frac{1}{8} \quad > \quad \cdots$$

We can therefore conclude that the value of S_n is approaching a maximum value of 1; it is converging to 1.

Step 3: Write conclusion mathematically

We can conclude that the sum of the series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

gets closer to 1 ($S_n \to 1$) as the number of terms approaches infinity ($n \to \infty$), therefore the series converges.

$$\sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^i = 1$$

We express the sum of an infinite number of terms of a series as

$$S_\infty = \sum_{i=1}^{\infty} T_i$$

Convergence and divergence

If the sum of a series gets closer and closer to a certain value as we increase the number of terms in the sum, we say that the series converges. In other words, there is a limit to the sum of a converging series. If a series does not converge, we say that it diverges. The sum of an infinite series usually tends to infinity, but there are some special cases where it does not.

Exercise 1 – 10: Convergent and divergent series

For each of the general terms below:

- Determine if it forms an arithmetic or geometric series.
- Calculate S_1, S_2, S_{10} and S_{100}.
- Determine if the series is convergent or divergent.

1. $T_n = 2n$
2. $T_n = (-n)$
3. $T_n = (\frac{2}{3})^n$
4. $T_n = 2^n$
5. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 287J 2. 287K 3. 287M 4. 287N

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Note the following:

- An arithmetic series never converges: as n tends to infinity, the series will always tend to positive or negative infinity.
- Some geometric series converge (have a limit) and some diverge (as n tends to infinity, the series does not tend to any limit or it tends to infinity).

Infinite geometric series EMCF4

There is a simple test for determining whether a geometric series converges or diverges; if $-1 < r < 1$, then the infinite series will converge. If r lies outside this interval, then the infinite series will diverge.

Test for convergence:

- If $-1 < r < 1$, then the infinite geometric series converges.
- If $r < -1$ or $r > 1$, then the infinite geometric series diverges.

We derive the formula for calculating the value to which a geometric series converges as follows:

$$S_n = \sum_{i=1}^{n} ar^{i-1} = \frac{a(1 - r^n)}{1 - r}$$

Now consider the behaviour of r^n for $-1 < r < 1$ as n becomes larger.

Let $r = \frac{1}{2}$:

$$n = 1 : r^n = r^1 = \left(\frac{1}{2}\right)^1 = \frac{1}{2}$$
$$n = 2 : r^n = r^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} < \frac{1}{2}$$
$$n = 3 : r^n = r^3 = \left(\frac{1}{2}\right)^3 = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8} < \frac{1}{4}$$

Since r is in the range $-1 < r < 1$, we see that r^n gets closer to 0 as n gets larger. Therefore $(1 - r^n)$ gets closer to 1.

Therefore,

$$\begin{aligned} S_n &= \frac{a(1 - r^n)}{1 - r} \\ \text{If } -1 < r < 1, \quad &\text{then } r^n \to 0 \text{ as } n \to \infty \\ \therefore S_\infty &= \frac{a(1 - 0)}{1 - r} \\ &= \frac{a}{1 - r} \end{aligned}$$

The sum of an infinite geometric series is given by the formula

$$\therefore S_\infty = \sum_{i=1}^{\infty} ar^{i-1} = \frac{a}{1 - r} \qquad (-1 < r < 1)$$

where

- a is the first term of the series;
- r is the constant ratio.

Alternative notation:

$$\underbrace{S_n}_{n \to \infty} \to \frac{a}{1 - r} \quad \text{if } -1 < r < 1$$

In words: as the number of terms (n) tends to infinity, the sum of a converging geometric series (S_n) tends to the value $\frac{a}{1-r}$.

See video: 287P at www.everythingmaths.co.za

Worked example 15: Sum to infinity of a geometric series

QUESTION

Given the series $18 + 6 + 2 + \cdots$. Find the sum to infinity if it exists.

SOLUTION

Step 1: Determine the value of r

We need to know the value of r to determine whether the series converges or diverges.

$$\begin{aligned} \frac{T_2}{T_1} &= \frac{6}{18} \\ &= \frac{1}{3} \\ \frac{T_3}{T_2} &= \frac{2}{6} \\ &= \frac{1}{3} \\ \therefore r &= \frac{1}{3} \end{aligned}$$

Since $-1 < r < 1$, we can conclude that this is a convergent geometric series.

Step 2: Determine the sum to infinity

Write down the formula for the sum to infinity and substitute the known values:

$$a = 18; \qquad r = \frac{1}{3}$$

$$\begin{aligned} S_\infty &= \frac{a}{1-r} \\ &= \frac{18}{1-\frac{1}{3}} \\ &= \frac{18}{\frac{2}{3}} \\ &= 18 \times \frac{3}{2} \\ &= 27 \end{aligned}$$

As n tends to infinity, the sum of this series tends to 27; no matter how many terms are added together, the value of the sum will never be greater than 27.

Worked example 16: Using the sum to infinity to convert recurring decimals to fractions

QUESTION

Use two different methods to convert the recurring decimal $0,\dot{5}$ to a proper fraction.

SOLUTION

Step 1: Convert the recurring decimal to a fraction using equations

$$\begin{aligned} \text{Let } x &= 0{,}\dot{5} \\ \therefore x &= 0{,}555\ldots\ldots(1) \\ 10x &= 5{,}55\ldots\ldots(2) \\ (2)-(1):\quad 9x &= 5 \\ \therefore x &= \frac{5}{9} \end{aligned}$$

Step 2: Convert the recurring decimal to a fraction using the sum to infinity

$$0{,}\dot{5} = 0{,}5 + 0{,}05 + 0{,}005 + \ldots$$
$$\text{or} \quad 0{,}\dot{5} = \frac{5}{10} + \frac{5}{100} + \frac{5}{1000} + \ldots$$

This is a geometric series with $r = 0{,}1 = \frac{1}{10}$. And since $-1 < r < 1$, we can conclude that the series is convergent.

$$\begin{aligned} S_\infty &= \frac{a}{1-r} \\ &= \frac{\frac{5}{10}}{1 - \frac{1}{10}} \\ &= \frac{\frac{5}{10}}{\frac{9}{10}} \\ &= \frac{5}{9} \end{aligned}$$

Worked example 17: Sum to infinity

QUESTION

Determine the possible values of a and r if

$$\sum_{n=1}^{\infty} ar^{n-1} = 5$$

SOLUTION

Step 1: Write down the sum to infinity formula and substitute known values

$$\begin{aligned} S_\infty &= \frac{a}{1-r} \\ \therefore 5 &= \frac{a}{1-r} \\ a &= 5(1-r) \\ \therefore a &= 5 - 5r \\ \text{And } 5r &= 5 - a \\ \therefore r &= \frac{5-a}{5} \end{aligned}$$

Step 2: Apply the condition for convergence to determine possible values of a

For a series to converge: $-1 < r < 1$

$$\begin{aligned} -1 < &\, r < 1 \\ -1 < &\, \frac{5-a}{5} < 1 \\ -5 < &\, 5 - a < 5 \\ -10 < &\, -a < 0 \\ 0 < &\, a < 10 \end{aligned}$$

Step 3: Write the final answer

For the series to converge, $0 < a < 10$ and $-1 < r < 1$.

Exercise 1 – 11:

1. What value does $\left(\frac{2}{5}\right)^n$ approach as n tends towards ∞?
2. Find the sum to infinity of the geometric series $3 + 1 + \frac{1}{3} + \frac{1}{9} + \cdots$
3. Determine for which values of x, the geometric series $2 + \frac{2}{3}(x+1) + \frac{2}{9}(x+1)^2 + \cdots$ will converge.
4. The sum to infinity of a geometric series with positive terms is $4\frac{1}{6}$ and the sum of the first two terms is $2\frac{2}{3}$. Find a, the first term, and r, the constant ratio between consecutive terms.
5. Use the sum to infinity to show that $0{,}\dot{9} = 1$.
6. A shrub 110 cm high is planted in a garden. At the end of the first year, the shrub is 120 cm tall. Thereafter the growth of the shrub each year is half of it's growth in the previous year. Show that the height of the shrub will never exceed 130 cm. Draw a graph of the relationship between time and growth.

   [IEB, Nov 2003]
7. Find p:

$$\sum_{k=1}^{\infty} 27p^k = \sum_{t=1}^{12} (24 - 3t)$$

8. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 287Q 2. 287R 3. 287S 4. 287T 5. 287V 6. 287W
7. 287X

1.7 Summary EMCF5

Arithmetic sequence

- common difference (d) between any two consecutive terms: $d = T_n - T_{n-1}$
- general form: $a + (a + d) + (a + 2d) + \cdots$
- general formula: $T_n = a + (n - 1)d$
- graph of the sequence lies on a straight line

Quadratic sequence

- common second difference between any two consecutive terms
- general formula: $T_n = an^2 + bn + c$
- graph of the sequence lies on a parabola

Geometric sequence

- constant ratio (r) between any two consecutive terms: $r = \frac{T_n}{T_{n-1}}$
- general form: $a + ar + ar^2 + \cdots$
- general formula: $T_n = ar^{n-1}$
- graph of the sequence lies on an exponential curve

Sigma notation

$$\sum_{k=1}^{n} T_k$$

Sigma notation is used to indicate the sum of the terms given by T_k, starting from $k = 1$ and ending at $k = n$.

Series

- the sum of certain numbers of terms in a sequence
- arithmetic series:
  - $S_n = \frac{n}{2}[a + l]$
  - $S_n = \frac{n}{2}[2a + (n - 1)d]$
- geometric series:
  - $S_n = \frac{a(1-r^n)}{1-r}$ if $r < 1$
  - $S_n = \frac{a(r^n-1)}{r-1}$ if $r > 1$

Sum to infinity

A convergent geometric series, with $-1 < r < 1$, tends to a certain fixed number as the number of terms in the sum tends to infinity.

$$S_\infty = \sum_{n=1}^{\infty} T_n = \frac{a}{1 - r}$$

Exercise 1 – 12: End of chapter exercises

1. Is $1 + 2 + 3 + 4 + \cdots$ an example of a *finite series* or an *infinite series*?
2. A new soccer competition requires each of 8 teams to play every other team once.
   a) Calculate the total number of matches to be played in the competition.
   b) If each of n teams played each other once, determine a formula for the total number of matches in terms of n.
3. Calculate:
$$\sum_{k=2}^{6} 3\left(\frac{1}{3}\right)^{k+2}$$
4. The first three terms of a convergent geometric series are: $x + 1; x - 1; 2x - 5$.
   a) Calculate the value of x, ($x \neq 1$ or 1).
   b) Sum to infinity of the series.
5. Write the sum of the first twenty terms of the following series in $\sum$ notation.
$$6 + 3 + \frac{3}{2} + \frac{3}{4} + \cdots$$
6. Determine:
$$\sum_{k=1}^{\infty} 12\left(\frac{1}{5}\right)^{k-1}$$
7. A man was injured in an accident at work. He receives a disability grant of R 4800 in the first year. This grant increases with a fixed amount each year.
   a) What is the annual increase if he received a total of R $143\,500$ over 20 years?
   b) His initial annual expenditure is R 2600, which increases at a rate of R 400 per year. After how many years will his expenses exceed his income?
8. The length of the side of a square is 4 units. This square is divided into 4 equal, smaller squares. One of the smaller squares is then divided into four equal, even smaller squares. One of the even smaller squares is divided into four, equal squares. This process is repeated indefinitely. Calculate the sum of the areas of all the squares.
9. Thembi worked part-time to buy a Mathematics book which costs R $29{,}50$. On 1 February she saved R $1{,}60$, and every day saves 30 cents more than she saved the previous day. So, on the second day, she saved R $1{,}90$, and so on. After how many days did she have enough money to buy the book?
10. A plant reaches a height of 118 mm after one year under ideal conditions in a greenhouse. During the next year, the height increases by 12 mm. In each successive year, the height increases by $\frac{5}{8}$ of the previous year's growth. Show that the plant will never reach a height of more than 150 mm.
11. Calculate the value of n if:
$$\sum_{a=1}^{n} (20 - 4a) = -20$$

12. Michael saved R 400 during the first month of his working life. In each subsequent month, he saved 10% more than what he had saved in the previous month.

    a) How much did he save in the seventh working month?
    b) How much did he save all together in his first 12 working months?

13. The Cape Town High School wants to build a school hall and is busy with fundraising. Mr. Manuel, an ex-learner of the school and a successful politician, offers to donate money to the school. Having enjoyed mathematics at school, he decides to donate an amount of money on the following basis. He sets a mathematical quiz with 20 questions. For the correct answer to the first question (any learner may answer), the school will receive R 1, for a correct answer to the second question, the school will receive R 2, and so on. The donations $1; 2; 4; \ldots$ form a geometric sequence. Calculate, to the nearest Rand:

    a) The amount of money that the school will receive for the correct answer to the 20^{th} question.
    b) The total amount of money that the school will receive if all 20 questions are answered correctly.

14. The first term of a geometric sequence is 9, and the ratio of the sum of the first eight terms to the sum of the first four terms is $97 : 81$. Find the first three terms of the sequence, if it is given that all the terms are positive.
15. Given the geometric sequence: $6 + p; 10 + p; 15 + p$

    a) Determine p, $(p \neq -6 \text{ or } -10)$.
    b) Show that the constant ratio is $\frac{5}{4}$.
    c) Determine the tenth term of this sequence correct to one decimal place.

16. The second and fourth terms of a convergent geometric series are 36 and 16, respectively. Find the sum to infinity of this series, if all its terms are positive.
17. Evaluate:

$$\sum_{k=2}^{5} \frac{k\,(k+1)}{2}$$

18. $S_n = 4n^2 + 1$ represents the sum of the first n terms of a particular series. Find the second term.
19. Determine whether the following series converges for the given values of x. If it does converge, calculate the sum to infinity.

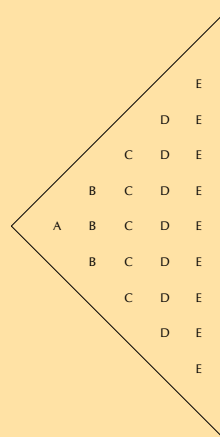
$$\sum_{p=1}^{\infty} (x+2)^p$$

    a) $x = -\frac{5}{2}$
    b) $x = -5$

20. Calculate:

$$\sum_{i=1}^{\infty} 5\left(4^{-i}\right)$$

21. The sum of the first p terms of a sequence is $p\,(p+1)$. Find the tenth term.
22. The powers of 2 are removed from the following set of positive integers
    $1; 2; 3; 4; 5; 6; \ldots; 1998; 1999; 2000$
    Find the sum of remaining integers.

23. Observe the pattern below:

a) If the pattern continues, find the number of letters in the column containing M's.

b) If the total number of letters in the pattern is 361, which letter will be in the last column.

24. Write $0{,}5\dot{7}$ as a proper fraction.

25. Given:

$$f(x) = \sum_{p=1}^{\infty} \frac{(1+x)^p}{1-x}$$

a) For which values of x will $f(x)$ converge?

b) Determine the value of $f\left(-\frac{1}{2}\right)$.

26. From the definition of a geometric sequence, deduce a formula for calculating the sum of n terms of the series

$$a^2 + a^4 + a^6 + \cdots$$

27. Calculate the tenth term of the series if $S_n = 2n + 3n^2$.

28. A theatre is filling up at a rate of 4 people in the first minute, 6 people in the second minute, and 8 people in the third minute and so on. After 6 minutes the theatre is half full. After how many minutes will the theatre be full?
[IEB, Nov 2001]

29. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 287Y	2a. 287Z	2b. 2882	3. 2883	4a. 2884	4b. 2885
5. 2886	6. 2887	7a. 2888	7b. 2889	8. 288B	9. 288C
10. 288D	11. 288F	12a. 288G	12b. 288H	13a. 288J	13b. 288K
14. 288M	15a. 288N	15b. 288P	15c. 288Q	16. 288R	17. 288S
18. 288T	19a. 288V	19b. 288W	20. 288X	21. 288Y	22. 288Z
23a. 2892	23b. 2893	24. 2894	25. 2895	26. 2896	27. 2897
28. 2898					

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CHAPTER

Functions

2.1 Revision EMCF6

In previous grades we learned about the characteristics of linear, quadratic, hyperbolic and exponential functions. In this chapter we will demonstrate the ability to work with various types of functions and relations including inverses. In particular, we will look at the graphs of the inverses of:

Linear functions: $y = mx + c$ or $y = ax + q$

Quadratic functions: $y = ax^2$

Exponential functions: $y = b^x \quad (b > 0, b \neq 1)$

Worked example 1: Linear function

QUESTION

Draw a graph of $2y + x - 8 = 0$ and determine the significant characteristics of this linear function.

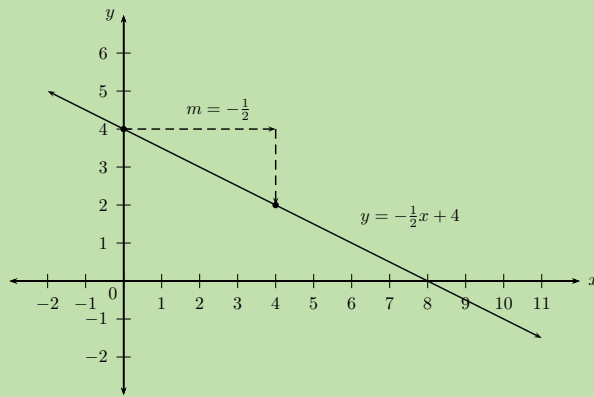
SOLUTION

Step 1: Write the equation in standard form $y = mx + c$

$$\begin{aligned} 2y + x - 8 &= 0 \\ 2y &= -x + 8 \\ \therefore y &= -\frac{1}{2}x + 4 \\ \therefore m &= -\frac{1}{2} \\ \text{And } c &= 4 \end{aligned}$$

Step 2: Draw the straight line graph

To draw the straight line graph we can use the gradient-intercept method:

$$\begin{aligned} y - \text{intercept}: \ & (0;4) \\ m &= -\frac{1}{2} \end{aligned}$$

Alternative method: we can also determine and plot the x- and y-intercepts as follows:

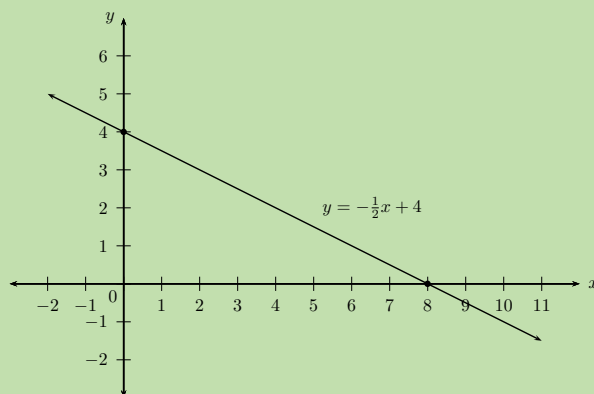
For the y-intercept, let $x = 0$:

$$\begin{aligned} y &= -\frac{1}{2}(0) + 4 \\ \therefore y &= 0 + 4 \\ &= 4 \end{aligned}$$

This gives the point $(0; 4)$.

For the x-intercept, let $y = 0$:

$$\begin{aligned} 0 &= -\frac{1}{2}x + 4 \\ \frac{1}{2}x &= 4 \\ \therefore x &= 8 \end{aligned}$$

This gives the point $(8; 0)$.

Step 3: Determine the characteristics

Gradient: $-\frac{1}{2}$

Intercepts: $(0; 4)$ and $(8; 0)$

Domain: $\{x : x \in \mathbb{R}\}$

Range: $\{y : y \in \mathbb{R}\}$

Decreasing function: as x increases, y decreases.

Worked example 2: Quadratic function

QUESTION

Write the quadratic function $2y - x^2 + 4 = 0$ in standard form. Draw a graph of the function and state the significant characteristics.

SOLUTION

Step 1: Write the equation in standard form $y = ax^2 + bx + c$

$$\begin{aligned} 2y - x^2 + 4 &= 0 \\ 2y &= x^2 - 4 \\ y &= \frac{1}{2}x^2 - 2 \end{aligned}$$

Therefore, we see that:

$$a = \frac{1}{2}; \qquad b = 0; \qquad c = -2$$

Step 2: Draw a graph of the parabola

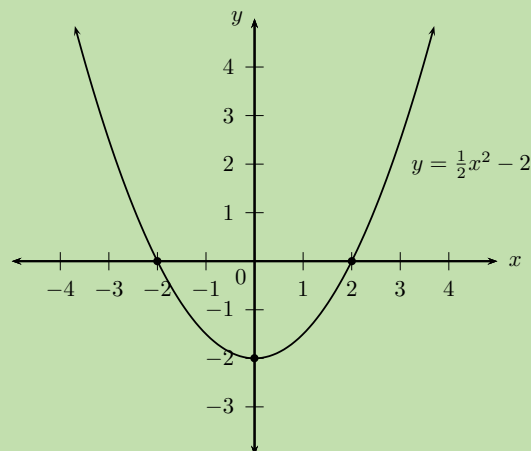
For the y-intercept, let $x = 0$:

$$\begin{aligned} y &= \frac{1}{2}(0)^2 - 2 \\ \therefore y &= 0 - 2 \\ &= -2 \end{aligned}$$

This gives the point $(0; -2)$.

For the x-intercept, let $y = 0$:

$$\begin{aligned} 0 &= \frac{1}{2}x^2 - 2 \\ 0 &= x^2 - 4 \\ 0 &= (x-2)(x+2) \\ \therefore x &= -2 \text{ or } x = 2 \end{aligned}$$

This gives the points $(-2; 0)$ and $(2; 0)$.

Step 3: State the significant characteristics

Shape: $a > 0$, therefore the graph is a "smile".

Intercepts: $(-2; 0), (2; 0)$ and $(0; -2)$

Turning point: $(0; -2)$

Axes of symmetry: $x = -\frac{b}{2a} = -\frac{0}{2\left(\frac{1}{2}\right)} = 0$

Domain: $\{x : x \in \mathbb{R}\}$

Range: $\{y : y \geq -2, y \in \mathbb{R}\}$

The function is decreasing for $x < 0$ and increasing for $x > 0$.

Worked example 3: Exponential function

QUESTION

Draw the graphs of $f(x) = 2^x$ and $g(x) = \left(\frac{1}{2}\right)^x$ on the same set of axes and compare the two functions.

SOLUTION

Step 1: Examine the functions and determine the information needed to draw the graphs

Consider the function: $f(x) = 2^x$

$$\begin{aligned} \text{If } y = 0: \quad 2^x &= 0 \\ \text{But } 2^x &\neq 0 \\ \therefore \text{ no solution} \\ \text{If } x = 0: \quad 2^0 &= 1 \end{aligned}$$

This gives the point $(0; 1)$.

Asymptotes: $f(x) = 2^x$ has a horizontal asymptote, the line $y = 0$, which is the x-axis.

x	-2	-1	0	1	2
$f(x)$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4

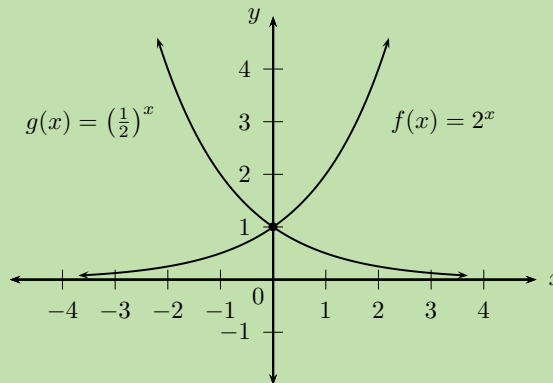
Consider the function: $g(x) = \left(\frac{1}{2}\right)^x$

$$\begin{aligned} \text{If } y = 0: \quad \left(\frac{1}{2}\right)^x &= 0 \\ \text{But } \left(\frac{1}{2}\right)^x &\neq 0 \\ \therefore \text{ no solution} \\ \text{If } x = 0: \quad \left(\frac{1}{2}\right)^0 &= 1 \end{aligned}$$

This gives the point $(0; 1)$.

Asymptotes: $g(x) = \left(\frac{1}{2}\right)^x$ also has a horizontal asymptote at $y = 0$.

x	-2	-1	0	1	2
$g(x)$	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$

Step 2: Draw the graphs of the exponential functions

Step 3: State the significant characteristics

Symmetry: f and g are symmetrical about the y-axis.

Domain of f and g: $\{x : x \in \mathbb{R}\}$

Range of f and g: $\{y : y > 0, y \in \mathbb{R}\}$

The function g decreases as x increases and function f increases as x increases. The two graphs intersect at the point $(0; 1)$.

Exercise 2 – 1: Revision

1. Sketch the graphs on the same set of axes and determine the following for each function:

   - Intercepts
   - Turning point
   - Axes of symmetry
   - Domain and range
   - Maximum and minimum values

   a) $f(x) = 3x^2$ and $g(x) = -x^2$
   b) $j(x) = -\frac{1}{5}x^2$ and $k(x) = -5x^2$
   c) $h(x) = 2x^2 + 4$ and $l(x) = -2x^2 - 4$

2. Given $f(x) = -3x - 6$ and $g(x) = mx + c$. Determine the values of m and c if $g \parallel f$ and g passes through the point $(1; 2)$. Sketch both functions on the same system of axes.
3. Given $m : \frac{x}{2} - \frac{y}{3} = 1$ and $n : -\frac{y}{3} = 1$. Determine the x- and y-intercepts and sketch both graphs on the same system of axes.
4. Given $p(x) = 3^x, q(x) = 3^{-x}$ and $r(x) = -3^x$.

   a) Sketch p, q and r on the same system of axes.
   b) For each of the functions, determine the intercepts, asymptotes, domain and range.

5. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 2899 1b. 289B 1c. 289C 2. 289D 3. 289F 4a. 289G
4b. 289H

2.2 Functions and relations EMCF7

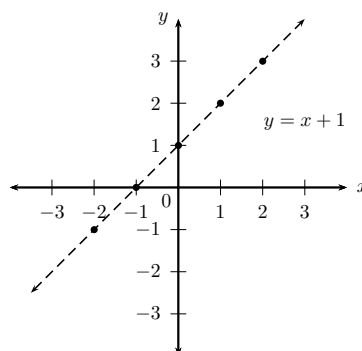
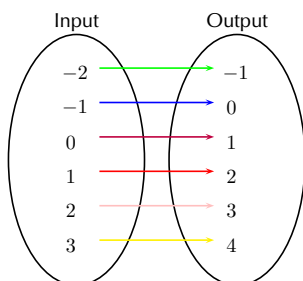
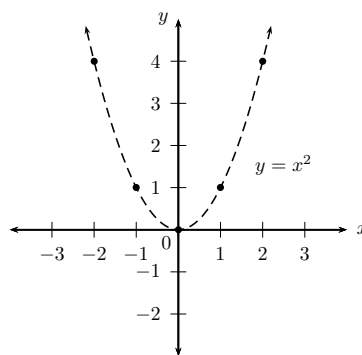
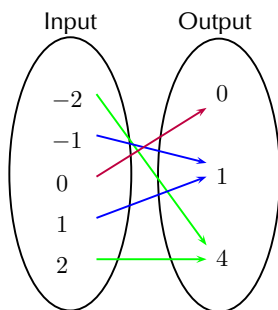
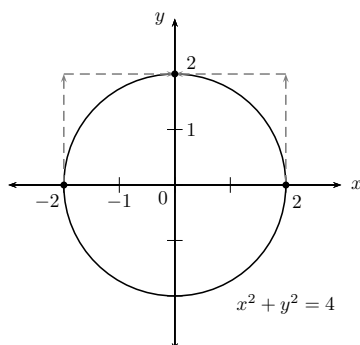
DEFINITION: *Relation*

A rule which associates each element of set (A) with at least one element in set (B).

DEFINITION: *Function*

A rule which uniquely associates elements of one set (A) with the elements of another set (B); each element in set (A) maps to only one element in set (B).

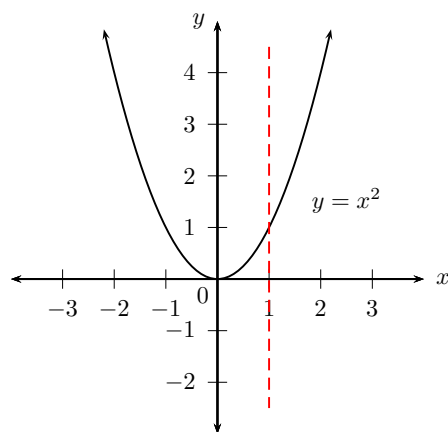
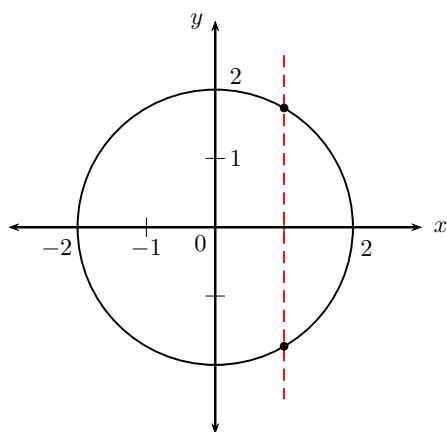
Functions can be one-to-one relations or many-to-one relations. A many-to-one relation associates two or more values of the independent (input) variable with a single value of the dependent (output) variable. The domain is the set of values to which the rule is applied (A) and the range is the set of values (also called the images or function values) determined by the rule.

Example of a one-to-one function: $y = x + 1$

Example of a many-to-one function: $y = x^2$

However, some very common mathematical constructions are not functions. For example, consider the relation $x^2 + y^2 = 4$. This relation describes a circle of radius 2 centred at the origin. If we let $x = 0$, we see that $y^2 = 4$ and thus either $y = 2$ or $y = -2$. This is a one-to-many relation because a single x-value relates to two different y-values. Therefore $x^2 + y^2 = 4$ is not a function.

Vertical line test

Given the graph of a relation, there is a simple test for whether or not the relation is a function. This test is called the vertical line test. If it is possible to draw any vertical line (a line of constant x) which crosses the graph of the relation more than once, then the relation is not a function. If more than one intersection point exists, then the intersections correspond to multiple values of y for a single value of x (one-to-many).

If any vertical line cuts the graph only once, then the relation is a function (one-to-one or many-to-one).

The red vertical line cuts the circle twice and therefore the circle is not a function.

The red vertical line only cuts the parabola once and therefore the parabola is a function.

Exercise 2 – 2: Identifying functions

1. Consider the graphs given below and determine whether or not they are functions:

a)

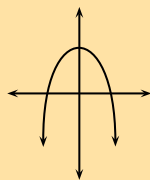
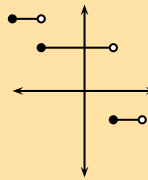
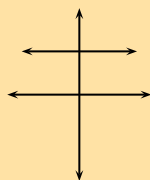
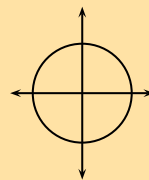
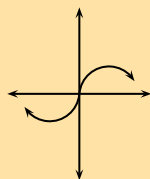
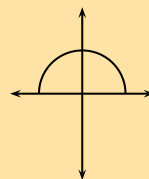
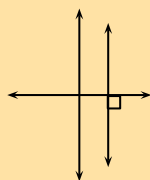
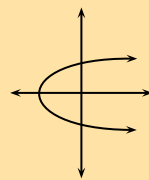
b)

c)

d)

e)

f)

g)

h)

2. Sketch the following and determine whether or not they are functions:

   a) $x + y = 4$

   b) $y = \frac{x}{4}$

   c) $y = 2^x$

   d) $x^2 + y^2 = 9$

   e) $y = \tan x$

3. The table below gives the average per capita income, d, in a region of the country as a function of u, the percentage of unemployed people. Write down an equation to show that the average income is a function of the percentage of unemployed people.

u	1	2	3	4
d	22 500	22 000	21 500	21 000

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 289J 1b. 289K 1c. 289M 1d. 289N 1e. 289P 1f. 289Q
1g. 289R 1h. 289S 2a. 289T 2b. 289V 2c. 289W 2d. 289X
2e. 289Y 3. 289Z

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Function notation

For the function $y = f(x)$, y is the dependent variable, because the value of y (output) depends on the value of x (input). We say x is the independent variable, since we can choose x to be any number. Similarly, if $g(t) = 2t + 1$, then t is the independent variable and g is the function name.

- If $h(x) = 3x - 5$ and we need to determine when $h(x) = 3$, then we solve for the value of x such that:

$$\begin{aligned} h(x) &= 3x - 5 \\ 3 &= 3x - 5 \\ 8 &= 3x \\ \therefore x &= \frac{8}{3} \end{aligned}$$

- If $h(x) = 3x - 5$ and we need to determine $h(3)$, then we calculate the value for $h(x)$ when $x = 3$:

$$\begin{aligned} h(x) &= 3x - 5 \\ h(3) &= 3(3) - 5 \\ &= 4 \end{aligned}$$

2.3 Inverse functions EMCF8

An inverse function is a function which does the "reverse" of a given function. More formally, if f is a function with domain X, then f^{-1} is its inverse function if and only if $f^{-1}(f(x)) = x$ for every $x \in X$.

$y = f(x)$: indicates a function

$y_1 = f(x_1)$: indicates we must substitute a specific x_1 value into the function to get the corresponding y_1 value

$f^{-1}(y) = x$: indicates the inverse function

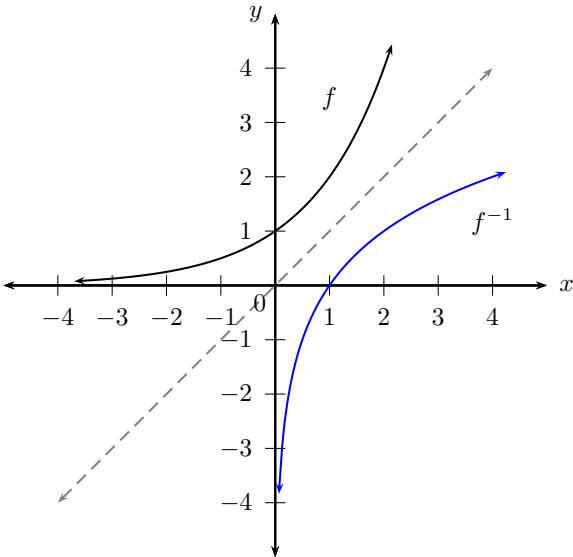
$f^{-1}(y_1) = x_1$: indicates we must substitute a specific y_1 value into the inverse to return the specific x_1 value

A function must be a one-to-one relation if its inverse is to be a function. If a function f has an inverse function f^{-1}, then f is said to be invertible.

Given the function $f(x)$, we determine the inverse $f^{-1}(x)$ by:

- interchanging x and y in the equation;
- making y the subject of the equation;
- expressing the new equation in function notation.

Note: if the inverse is not a function then it cannot be written in function notation. For example, the inverse of $f(x) = 3x^2$ cannot be written as $f^{-1}(x) = \pm\sqrt{\frac{1}{3}x}$ as it is not a function. We write the inverse as $y = \pm\sqrt{\frac{1}{3}x}$ and conclude that f is not invertible.

If we represent the function f and the inverse function f^{-1} graphically, the two graphs are reflected about the line $y = x$. Any point on the line $y = x$ has x- and y-coordinates with the same numerical value, for example $(-3; -3)$ and $\left(\frac{4}{5}; \frac{4}{5}\right)$. Therefore interchanging the x- and y-values makes no difference.

This diagram shows an exponential function (black graph) and its inverse (blue graph) reflected about the line $y = x$ (grey line).

Important: for f^{-1}, the superscript -1 is not an exponent. It is the notation for indicating the inverse of a function. Do not confuse this with exponents, such as $\left(\frac{1}{2}\right)^{-1}$ or $3 + x^{-1}$.

Be careful not to confuse the inverse of a function and the reciprocal of a function:

Inverse	**Reciprocal**
$f^{-1}(x)$	$[f(x)]^{-1} = \frac{1}{f(x)}$
$f(x)$ and $f^{-1}(x)$ symmetrical about $y = x$	$f(x) \times \frac{1}{f(x)} = 1$
Example:	Example:
$g(x) = 5x \therefore g^{-1}(x) = \frac{x}{5}$	$g(x) = 5x \therefore \frac{1}{g(x)} = \frac{1}{5x}$

See video: 28B2 at www.everythingmaths.co.za

2.4 Linear functions EMCF9

Inverse of the function $y = ax + q$ EMCFB

Worked example 4: Inverse of the function $y = ax + q$

QUESTION

Determine the inverse function of $p(x) = -3x + 1$ and sketch the graphs of $p(x)$ and $p^{-1}(x)$ on the same system of axes.

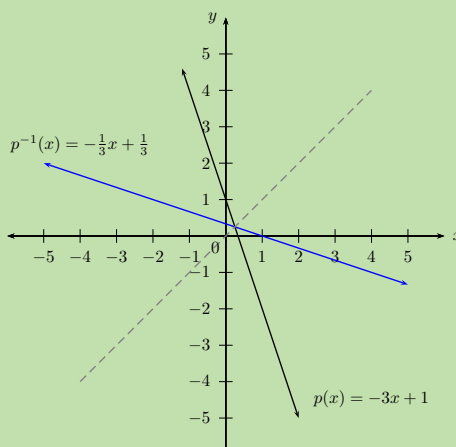
SOLUTION

Step 1: Determine the inverse of the given function

- Interchange x and y in the equation.
- Make y the subject of the new equation.
- Express the new equation in function notation.

$$\begin{aligned} \text{Let } y &= -3x + 1 \\ \text{Interchange } x \text{ and } y: \quad x &= -3y + 1 \\ x - 1 &= -3y \\ -\frac{1}{3}(x - 1) &= y \\ \therefore y &= -\frac{x}{3} + \frac{1}{3} \end{aligned}$$

Therefore, $p^{-1}(x) = -\frac{x}{3} + \frac{1}{3}$.

Step 2: Sketch the graphs of the same system of axes

The graph of $p^{-1}(x)$ is the reflection of $p(x)$ about the line $y = x$. This means that every point on the graph of $p(x)$ has a mirror image on the graph of $p^{-1}(x)$.

To determine the inverse function of $y = ax + q$:

(1)	Interchange x and y :	$x = ay + q$
(2)	Make y the subject of the equation :	$x - q = ay$
		$\frac{x}{a} - \frac{q}{a} = \frac{ay}{a}$
		$\therefore y = \frac{1}{a}x - \frac{q}{a}$

Therefore the inverse of $y = ax + q$ is $y = \frac{1}{a}x - \frac{q}{a}$. If a linear function is invertible, then its inverse will also be linear.

Worked example 5: Inverses - domain, range and intercepts

QUESTION

Determine and sketch the inverse of the function $f(x) = 2x - 3$. State the domain, range and intercepts.

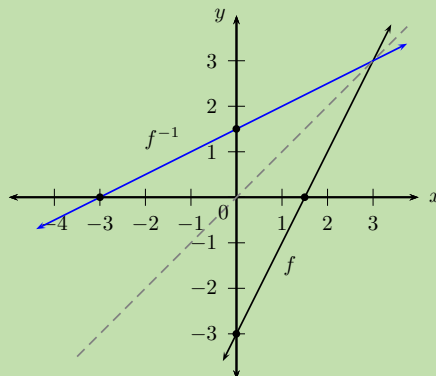
SOLUTION

Step 1: Determine the inverse of the given function

- Interchange x and y in the equation.
- Make y the subject of the new equation.
- Express the new equation in function notation.

$$\text{Let } y = 2x - 3$$
$$\text{Interchange } x \text{ and } y: \quad x = 2y - 3$$
$$x + 3 = 2y$$
$$\frac{1}{2}(x + 3) = y$$
$$\therefore y = \frac{x}{2} + \frac{3}{2}$$

Therefore, $f^{-1}(x) = \frac{x}{2} + \frac{3}{2}$.

Step 2: Sketch the graphs on the same system of axes

The graph of $f^{-1}(x)$ is the reflection of $f(x)$ about the line $y = x$.

Step 3: Determine domain, range and intercepts

$$\begin{aligned} \text{Domain of } f &: \{x : x \in \mathbb{R}\} \\ \text{Range of } f &: \{y : y \in \mathbb{R}\} \\ \text{Intercepts of } f &: (0; -3) \text{ and } \left(\frac{3}{2}; 0\right) \end{aligned}$$

$$\begin{aligned} \text{Domain of } f^{-1} &: \{x : x \in \mathbb{R}\} \\ \text{Range of } f^{-1} &: \{y : y \in \mathbb{R}\} \\ \text{Intercepts of } f^{-1} &: \left(0; \frac{3}{2}\right) \text{ and } (-3; 0) \end{aligned}$$

Notice that the intercepts of f and f^{-1} are mirror images of each other. In other words, the x- and y-values have "swapped" positions. This is true of every point on the two graphs.

Domain and range

For a function of the form $y = ax + q$, the domain is $\{x : x \in \mathbb{R}\}$ and the range is $\{y : y \in \mathbb{R}\}$. When a function is inverted the domain and range are interchanged. Therefore, the domain and range of the inverse of an invertible, linear function will be $\{x : x \in \mathbb{R}\}$ and $\{y : y \in \mathbb{R}\}$ respectively.

Intercepts

The general form of an invertible, linear function is $y = ax + q \ \ (a \neq 0)$ and its inverse is $y = \frac{1}{a}x - \frac{q}{a}$.

The y-intercept is obtained by letting $x = 0$:

$$\begin{aligned} y &= \frac{1}{a}(0) - \frac{q}{a} \\ y &= -\frac{q}{a} \end{aligned}$$

This gives the point $\left(0; -\frac{q}{a}\right)$.

The x-intercept is obtained by letting $y = 0$:

$$\begin{aligned} 0 &= \frac{1}{a}x - \frac{q}{a} \\ \frac{q}{a} &= \frac{1}{a}x \\ q &= x \end{aligned}$$

This gives the point $(q; 0)$.

It is interesting to note that if $f(x) = ax + q \ \ (a \neq 0)$, then $f^{-1}(x) = \frac{1}{a}x - \frac{q}{a}$ and the y-intercept of $f(x)$ is the x-intercept of $f^{-1}(x)$ and the x-intercept of $f(x)$ is the y-intercept of $f^{-1}(x)$.

See video: 28B3 at www.everythingmaths.co.za

Exercise 2 – 3: Inverse of the function $y = ax + q$

1. Given $f(x) = 5x + 4$, find $f^{-1}(x)$.
2. Consider the relation $f(x) = -3x - 7$.
   a) Is the relation a function? Explain your answer.
   b) Identify the domain and range.
   c) Determine $f^{-1}(x)$.
3. a) Sketch the graph of the function $f(x) = 3x - 1$ and its inverse on the same system of axes. Indicate the intercepts and the axis of symmetry of the two graphs.
   b) $T\left(\frac{4}{3}; 3\right)$ is a point on f and R is a point on f^{-1}. Determine the coordinates of R if R and T are symmetrical.
4. a) Explain why the line $y = x$ is an axis of symmetry for a function and its inverse.
   b) Will the line $y = -x$ be an axis of symmetry for a function and its inverse?
5. a) Given $f^{-1}(x) = -2x + 4$, determine $f(x)$.
   b) Calculate the intercepts of $f(x)$ and $f^{-1}(x)$.
   c) Determine the coordinates of T, the point of intersection of $f(x)$ and $f^{-1}(x)$.
   d) Sketch the graphs of f and f^{-1} on the same system of axes. Indicate the intercepts and point T on the graph.
   e) Is f^{-1} an increasing or decreasing function?
6. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 28B4 2a. 28B5 2b. 28B6 2c. 28B7 3. 28B8 4. 28B9
5. 28BB

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2.5 Quadratic functions EMCFC

Inverse of the function $y = ax^2$ EMCFD

Worked example 6: Inverse of the function $y = ax^2$

QUESTION

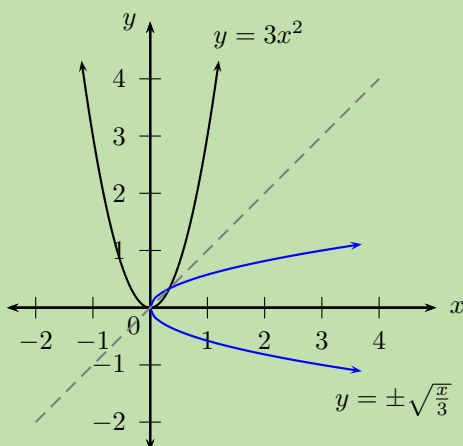
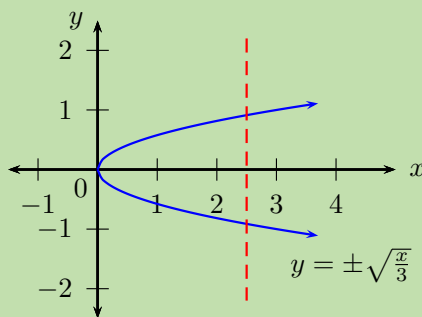
Determine the inverse of the quadratic function $h(x) = 3x^2$ and sketch both graphs on the same system of axes.

SOLUTION

Step 1: Determine the inverse of the given function $h(x)$

- Interchange x and y in the equation.
- Make y the subject of the new equation.

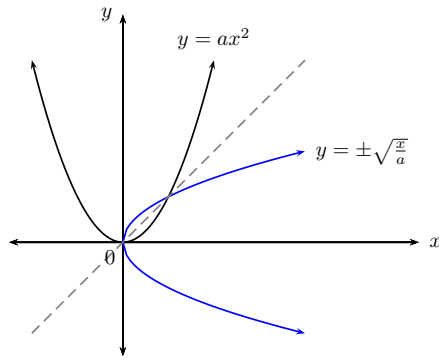
$$
\begin{aligned}
\text{Let } y &= 3x^2 \\
\text{Interchange } x \text{ and } y: \quad x &= 3y^2 \\
\frac{x}{3} &= y^2 \\
\therefore y &= \pm\sqrt{\frac{x}{3}} \qquad (x \geq 0)
\end{aligned}
$$

Step 2: Sketch the graphs on the same system of axes

Notice that the inverse does not pass the vertical line test and therefore is not a function.

To determine the inverse function of $y = ax^2$:

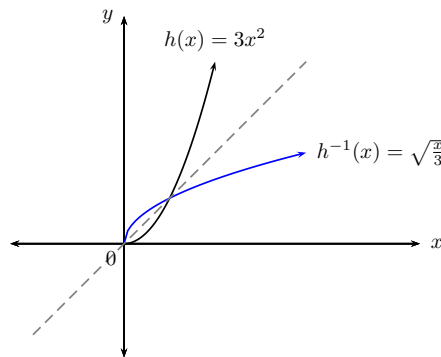
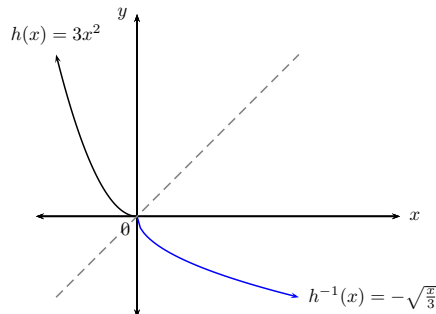
(1) Interchange x and y: $\quad x = ay^2$

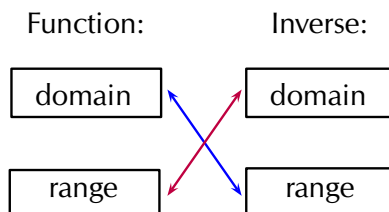
(2) Make y the subject of the equation: $\quad \frac{x}{a} = y^2$

$$\therefore y = \pm\sqrt{\frac{x}{a}} \qquad (x \geq 0)$$

The vertical line test shows that the inverse of a parabola is not a function. However, we can limit the domain of the parabola so that the inverse of the parabola is a function.

Domain and range

Consider the previous worked example $h(x) = 3x^2$ and its inverse $y = \pm\sqrt{\frac{x}{3}}$:

- If we restrict the domain of h so that $x \geq 0$, then $h^{-1}(x) = \sqrt{\frac{x}{3}}$ passes the vertical line test and is a function.

- If the restriction on the domain of h is $x \leq 0$, then $h^{-1}(x) = -\sqrt{\frac{x}{3}}$ would also be a function.

The domain of the function is equal to the range of the inverse. The range of the function is equal to the domain of the inverse.

Similarly, a restriction on the domain of the function results in a restriction on the range of the inverse and vice versa.

Worked example 7: Inverses - domain, range and restrictions

QUESTION

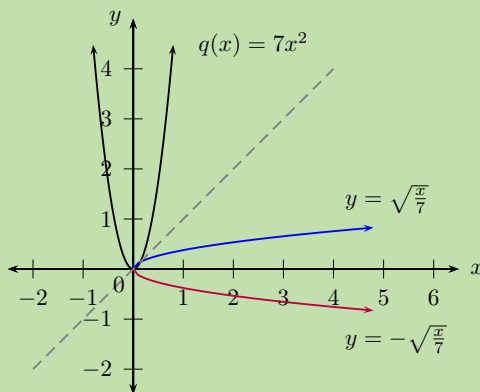
Determine the inverse of $q(x) = 7x^2$ and sketch both graphs on the same system of axes. Restrict the domain of q so that the inverse is a function.

SOLUTION

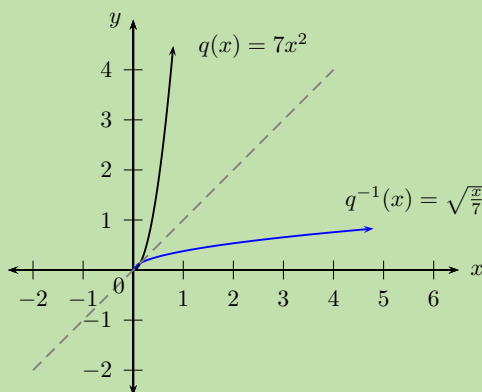
Step 1: Examine the function and determine the inverse

Determine the inverse of the function:

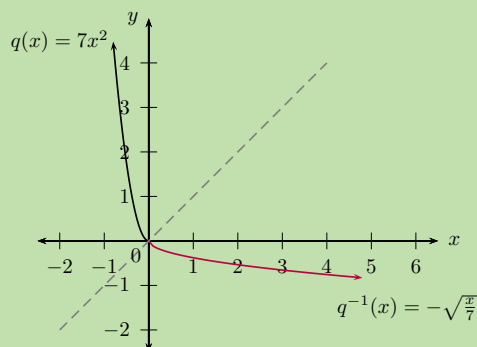
$$\begin{aligned} \text{Let } y &= 7x^2 \\ \text{Interchange } x \text{ and } y: \quad x &= 7y^2 \\ \frac{x}{7} &= y^2 \\ \therefore y &= \pm\sqrt{\frac{x}{7}} \qquad (x \geq 0) \end{aligned}$$

Step 2: Sketch both graphs on the same system of axes

Step 3: Determine the restriction on the domain

Option 1: Restrict the domain of q to $x \geq 0$ so that the inverse will also be a function $\left(q^{-1}\right)$. The restriction $x \geq 0$ on the domain of q will restrict the range of q^{-1} such that $y \geq 0$.

$$\begin{aligned} q: &\quad \text{domain } x \geq 0 \quad \text{range } y \geq 0 \\ q^{-1}: &\quad \text{domain } x \geq 0 \quad \text{range } y \geq 0 \end{aligned}$$

OR

Option 2: Restrict the domain of q to $x \leq 0$ so that the inverse will also be a function $\left(q^{-1}\right)$. The restriction $x \leq 0$ on the domain of q will restrict the range of q^{-1} such that $y \leq 0$.

$$q: \quad \text{domain } x \leq 0 \quad \text{range } y \geq 0$$
$$q^{-1}: \quad \text{domain } x \geq 0 \quad \text{range } y \leq 0$$

Worked example 8: Inverses - domain, range and restrictions

QUESTION

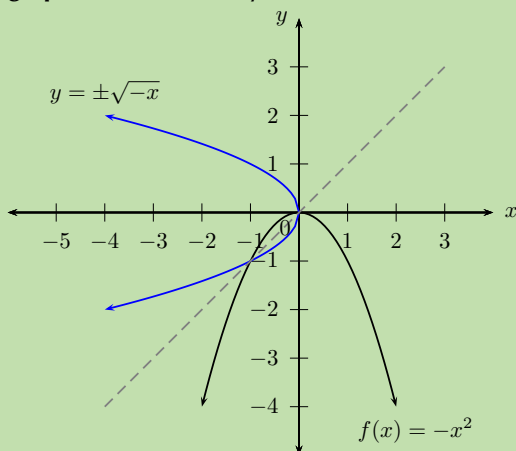
1. Determine the inverse of $f(x) = -x^2$.
2. Sketch both graphs on the same system of axes.
3. Restrict the domain of f so that its inverse is a function.

SOLUTION

Step 1: Determine the inverse of the function

$$\begin{aligned} \text{Let } y &= -x^2 \\ \text{Interchange } x \text{ and } y: \quad x &= -y^2 \\ -x &= y^2 \\ y &= \pm\sqrt{-x} \qquad (x \leq 0) \end{aligned}$$

Note: $\sqrt{-x}$ is only defined if $x \leq 0$.

Step 2: Sketch both graphs on the same system of axes

The inverse does not pass the vertical line test and is not a function.

Step 3: Determine the restriction on the domain

- If $f(x) = -x^2$, for $x \leq 0$:

$$\begin{aligned} f: &\quad \text{domain } x \leq 0 \quad \text{range } y \leq 0 \\ f^{-1}: &\quad \text{domain } x \leq 0 \quad \text{range } y \leq 0 \end{aligned}$$

- If $f(x) = -x^2$, for $x \geq 0$:

$$\begin{aligned} f: &\quad \text{domain } x \geq 0 \quad \text{range } y \leq 0 \\ f^{-1}: &\quad \text{domain } x \leq 0 \quad \text{range } y \geq 0 \end{aligned}$$

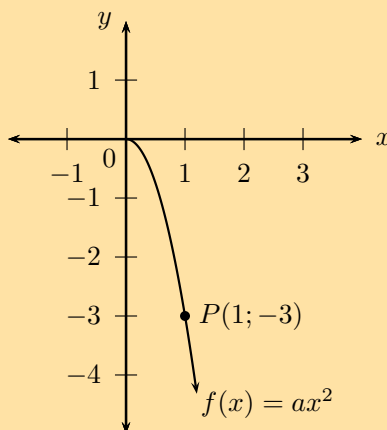
Exercise 2 – 4: Inverses - domain, range, intercepts, restrictions

1. Determine the inverse for each of the following functions:

   a) $y = \frac{3}{4}x^2$
   b) $4y - 8x^2 = 0$
   c) $x^2 + 5y = 0$
   d) $4y - 9 = (x + 3)(x - 3)$

2. Given the function $g(x) = \frac{1}{2}x^2$ for $x \geq 0$.

   a) Find the inverse of g.
   b) Draw g and g^{-1} on the same set of axes.
   c) Is g^{-1} a function? Explain your answer.
   d) State the domain and range for g and g^{-1}.
   e) Determine the coordinates of the point(s) of intersection of the function and its inverse.

3. Given the graph of the parabola $f(x) = ax^2$ with $x \geq 0$ and passing through the point $P(1; -3)$.

   a) Determine the equation of the parabola.
   b) State the domain and range of f.
   c) Give the coordinates of the point on f^{-1} that is symmetrical to the point P about the line $y = x$.

d) Determine the equation of f^{-1}.

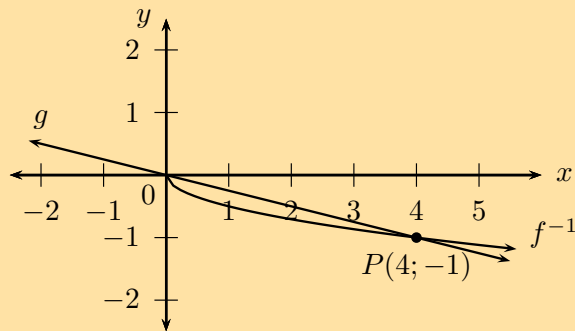
e) State the domain and range of f^{-1}.

f) Draw a graph of f^{-1}.

4. a) Determine the inverse of $h(x) = \frac{11}{5}x^2$.

   b) Sketch both graphs on the same system of axes.

   c) Restrict the domain of h so that the inverse is a function.

5. The diagram shows the graph of $g(x) = mx + c$ and $f^{-1}(x) = a\sqrt{x}, \ (x \geq 0)$. Both graphs pass through the point $P(4; -1)$.

   a) Determine the values of a, c and m.

   b) Give the domain and range of f^{-1} and g.

   c) For which values of x is $g(x) < f(x)$?

   d) Determine f.

   e) Determine the coordinates of the point(s) of intersection of g and f intersect.

6. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 28BC 1b. 28BD 1c. 28BF 1d. 28BG 2. 28BH 3. 28BJ

4. 28BK 5. 28BM

Worked example 9: Inverses - average gradient

QUESTION

Given: $h(x) = 2x^2, \quad x \geq 0$

1. Determine the inverse, h^{-1}.
2. Find the point where h and h^{-1} intersect.
3. Sketch h and h^{-1} on the same set of axes.
4. Use the sketch to determine if h and h^{-1} are increasing or decreasing functions.
5. Calculate the average gradient of h between the two points of intersection.

SOLUTION

Step 1: Determine the inverse of the function

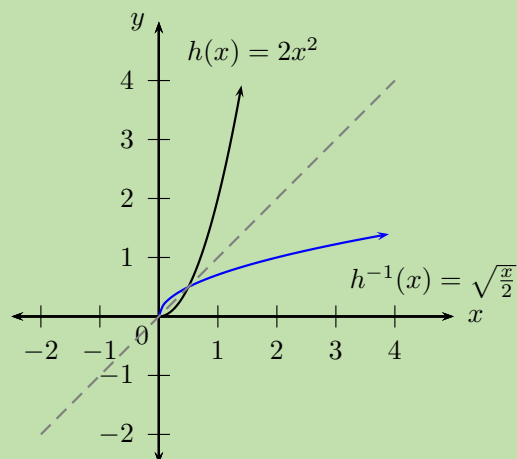
$$\begin{aligned}
\text{Let } y &= 2x^2 \qquad (x \geq 0) \\
\text{Interchange } x \text{ and } y: \quad x &= 2y^2 \qquad (y \geq 0) \\
\frac{x}{2} &= y^2 \\
y &= \sqrt{\frac{x}{2}} \qquad (x \geq 0, y \geq 0) \\
\therefore h^{-1}(x) &= \sqrt{\frac{x}{2}} \qquad (x \geq 0)
\end{aligned}$$

Step 2: Determine the point of intersection

$$\begin{aligned}
2x^2 &= \sqrt{\frac{x}{2}} \\
\left(2x^2\right)^2 &= \left(\sqrt{\frac{x}{2}}\right)^2 \\
4x^4 &= \frac{x}{2} \\
8x^4 &= x \\
8x^4 - x &= 0 \\
x(8x^3 - 1) &= 0 \\
\therefore x &= 0 \text{ or } 8x^3 - 1 = 0
\end{aligned}$$

$$\begin{aligned}
\text{If } x = 0, \quad y &= 0 \\
\text{If } 8x^3 - 1 &= 0 \\
8x^3 &= 1 \\
x^3 &= \frac{1}{8} \\
\therefore x &= \frac{1}{2} \\
\text{If } x = \frac{1}{2}, \quad y &= \frac{1}{2}
\end{aligned}$$

Therefore, this gives the points A$(0;0)$ and $B\left(\frac{1}{2};\frac{1}{2}\right)$.

Step 3: Sketch both graphs on the same system of axes

Step 4: Examine the graphs

From the graphs, we see that both h and h^{-1} pass the vertical line test and therefore are functions.

$$\begin{aligned} h: &\quad \text{as } x \text{ increases, } y \text{ also increases, therefore } h \text{ is an increasing function.} \\ h^{-1}: &\quad \text{as } x \text{ increases, } y \text{ also increases, therefore } h^{-1} \text{ is an increasing function.} \end{aligned}$$

Step 5: Calculate the average gradient

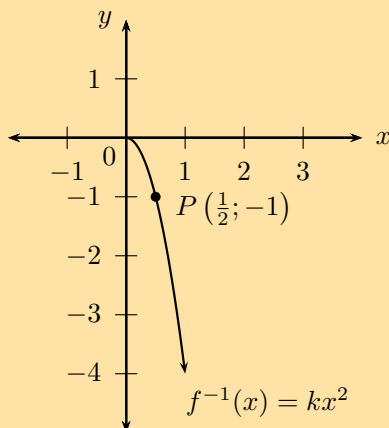
Calculate the average gradient of h between the points A$(0;0)$ and $B\left(\frac{1}{2};\frac{1}{2}\right)$.

$$\begin{aligned} \text{Average gradient: } &= \frac{y_B - y_A}{x_B - x_A} \\ &= \frac{\frac{1}{2} - 0}{\frac{1}{2} - 0} \\ &= 1 \end{aligned}$$

Note: this is also the average gradient of h^{-1} between the points A and B.

Exercise 2 – 5: Inverses - average gradient, increasing and decreasing functions

1. a) Sketch the graph of $y = x^2$ and label a point other than the origin on the graph.
   b) Find the equation of the inverse of $y = x^2$.
   c) Sketch the graph of the inverse on the same system of axes.
   d) Is the inverse a function? Explain your answer.
   e) $P(2;4)$ is a point on $y = x^2$. Determine the coordinates of Q, the point on the graph of the inverse which is symmetrical to P about the line $y = x$.
   f) Determine the average gradient between:
      i. the origin and P;
      ii. the origin and Q.

      Interpret the answers.
2. Given the function $f^{-1}(x) = kx^2$, $x \geq 0$, which passes through the point $P\left(\frac{1}{2};-1\right)$.

a) Find the value of k.

b) State the domain and range of f^{-1}.

c) Find the equation of f.

d) State the domain and range of f.

e) Sketch the graphs of f and f^{-1} on the same system of axes.

f) Is f an increasing or decreasing function?

3. Given: $g(x) = \frac{5}{2}x^2, \quad x \geq 0$.

   a) Find $g^{-1}(x)$.

   b) Calculate the point(s) where g and g^{-1} intersect.

   c) Sketch g and g^{-1} on the same set of axes.

   d) Use the sketch to determine if g and g^{-1} are increasing or decreasing functions.

   e) Calculate the average gradient of g^{-1} between the two points of intersection.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 28BN 2. 28BP 3. 28BQ

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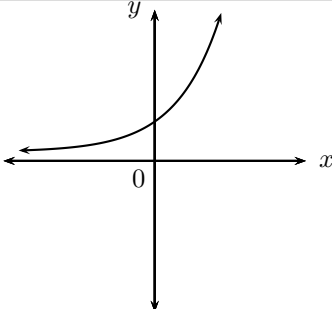
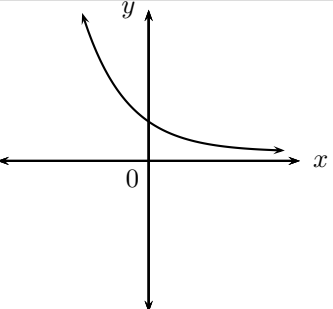
2.6 Exponential functions EMCFF

Revision of exponents

$$\text{base} \longleftarrow b^n \longrightarrow \text{exponent/index}$$

An exponent indicates the number of times a certain number (the base) is multiplied by itself. The exponent, also called the index or power, indicates the number of times the multiplication is repeated. For example, $10^3 = 10 \times 10 \times 10 = 1000$.

Graphs of the exponential function $f(x) = b^x$

The value of b affects the direction of the graph:

- If $b > 1$, $f(x)$ is an increasing function.
- If $0 < b < 1$, $f(x)$ is a decreasing function.
- If $b \leq 0$, $f(x)$ is not defined.

$y = b^x$	$b > 1$	$0 < b < 1$	$b \leq 0$
	(graph)	(graph)	Not defined

Inverse of the function $y = b^x$ EMCFG

Investigation: Determining the inverse

Function	Type of function	Inverse:	Inverse:
		interchange x and y	make y the subject
$y = \frac{x}{3} + 10$			
$y = \frac{x^2}{3}$			
$y = (10)^x$			
$y = \left(\frac{1}{3}\right)^x$			

Consider the exponential function

$$y = b^x$$

To determine the inverse of the exponential function, we interchange the x- and y-variables:

$$x = b^y$$

For straight line functions and parabolic functions, we could easily manipulate the inverse to make y the subject of the formula. For the inverse of an exponential function, however, y is the index and we do not know a method of solving for the index.

To resolve this problem, mathematicians defined the logarithmic function. The logarithmic function allows us to rewrite the expression $x = b^y$ with y as the subject of the formula:

$$y = \log_b x$$

This means that $x = b^y$ is the same as $y = \log_b x$ and both are the inverse of the exponential function $y = b^x$.

Logarithms EMCFH

DEFINITION: *Logarithm*

If $x = b^y$, then $y = \log_b(x)$, where $b > 0$, $b \neq 1$ and $x > 0$.

Note that the brackets around the number (x) are not compulsory, we use them to avoid confusion.

The logarithm of a number (x) with a certain base (b) is equal to the exponent (y), the value to which that certain base must be raised to equal the number (x).

For example, $\log_2(8)$ means the power of 2 that will give 8. Since $2^3 = 8$, we see that $\log_2(8) = 3$. Therefore the exponential form is $2^3 = 8$ and the logarithmic form is $\log_2 8 = 3$.

Restrictions on the definition of logarithms

Restriction:	Reason:
$b > 0$	If b is a negative number, then b^y will oscillate between: positive values if y is even negative values if y is odd
$b \neq 1$	Since $1^{\text{(any value)}} = 1$
$x > 0$	Since $(\text{positive number})^{\text{(any value)}} > 0$

Investigation: Exponential and logarithmic form

Discuss the following statements and determine whether they are true or false:

1. $p = a^n$ is the inverse of $p = \log_a n$.
2. $y = 2^x$ is a one-to-one function, therefore $y = \log_2 x$ is also a one-to-one function.
3. $x = \log_5 y$ is the inverse of $5^x = y$.
4. $k = b^t$ is the same as $t = \log_b k$.

To determine the inverse function of $y = b^x$:

(1)	Interchange x and y :	$x = b^y$
(2)	Make y the subject of the equation :	$y = \log_b x$

Therefore, if we have the exponential function $f(x) = b^x$, then the inverse is the logarithmic function $f^{-1}(x) = \log_b x$.

The "common logarithm" has a base 10 and can be written as $\log_{10} x = \log x$. In other words, the $\log$ symbol written without a base is interpreted as the logarithm to base 10. For example, $\log 25 = \log_{10} 25$.

See video: 28BR at www.everythingmaths.co.za

Worked example 10: Exponential form to logarithmic form

QUESTION

Write the following exponential expressions in logarithmic form and express each in words:

1. $5^2 = 25$
2. $10^{-3} = 0{,}001$
3. $p^x = q$

SOLUTION

Step 1: Determine the inverse of the given exponential expressions

Remember: $m = a^n$ is the same as $n = \log_a m$.

1. $2 = \log_5 25$
2. $-3 = \log_{10}(0{,}001)$
3. $x = \log_p q$

Step 2: Express in words

1. 2 is the power to which 5 must be raised to give the number 25.
2. -3 is the power to which 10 must be raised to give the decimal number $0{,}001$.
3. x is the power to which p must be raised to give q.

Worked example 11: Logarithmic form to exponential form

QUESTION

Write the following logarithmic expressions in exponential form:

1. $\log_2 128 = 7$
2. $-2 = \log_3 \left(\frac{1}{9}\right)$
3. $z = \log_w k$

SOLUTION

Step 1: Determine the inverse of the given logarithmic expressions

For $n = \log_a m$, we can write $m = a^n$.

1. $2^7 = 128$
2. $3^{-2} = \frac{1}{9}$
3. $w^z = k$

Exercise 2 – 6: Finding the inverse of $y = b^x$

1. Write the following in logarithmic form:

a) $16 = 2^4$
b) $3^{-5} = \frac{1}{243}$
c) $(1{,}7)^3 = 4{,}913$
d) $y = 2^x$
e) $q = 4^5$
f) $4 = y^g$
g) $9 = (x - 4)^p$
h) $3 = m^{(a+4)}$

2. Express each of the following logarithms in words and then write in exponential form:

a) $\log_2 32 = 5$
b) $\log \frac{1}{1000} = -3$
c) $\log 0{,}1 = -1$
d) $\log_d c = b$
e) $\log_5 1 = 0$
f) $\log_3 \frac{1}{81} = -4$
g) $\log 100$
h) $\log_{\frac{1}{2}} 16$

3. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 28BS 1b. 28BT 1c. 28BV 1d. 28BW 1e. 28BX 1f. 28BY
1g. 28BZ 1h. 28C2 2a. 28C3 2b. 28C4 2c. 28C5 2d. 28C6
2e. 28C7 2f. 28C8 2g. 28C9 2h. 28CB

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Logarithm bases EMCFJ

From the definition of a logarithm we know that the base of a logarithm must be a positive number and it cannot be equal to 1. The value of the base influences the value of the logarithm. For example, $\log_2 2$ is not the same as $\log 2$ and $\log_f 11$ is not the same as $\log_g 11$, $(f \neq g)$.

We often calculate the "common logarithm", which has a base 10 and can be written as $\log_{10} x = \log x$. For example, $\log 8 = \log_{10} 8$.

The "natural logarithm", which has a base e (an irrational number between $2{,}71$ and $2{,}72$), can be written as $\log_e x = \ln x$. For example, $\log_e 5 = \ln 5$.

Special logarithmic values

- $\log_a 1 = 0$

$$\begin{aligned} \text{Given the exponential form } a^n &= x \\ \text{we define the logarithmic function } \log_a x &= n \\ \text{So then for } a^0 &= 1 \\ \text{we can write } \log_a 1 &= 0 \end{aligned}$$

- $\log_a a = 1$

$$\begin{aligned} \text{From the general exponential form } a^n &= x \\ \text{we define the logarithmic function } \log_a x &= n \\ \text{Since } a^1 &= a \\ \text{we can write } \log_a a &= 1 \end{aligned}$$

In earlier grades, we used the following exponential laws for working with exponents:

- $a^m \times a^n = a^{m+n}$
- $\frac{a^m}{a^n} = a^{m-n}$
- $(ab)^n = a^n b^n$
- $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
- $(a^m)^n = a^{mn}$

where $a > 0$, $b > 0$ and $m, n \in \mathbb{Z}$.

The logarithmic laws are based on the exponential laws and make working with logarithms much easier.

Logarithmic laws:

- $\log_a x^b = b \log_a x \qquad (x > 0)$
- $\log_a x = \frac{\log_b x}{\log_b a} \qquad (b > 0$ and $b \neq 1)$
- $\log_a xy = \log_a x + \log_a y \qquad (x > 0$ and $y > 0)$
- $\log_a \frac{x}{y} = \log_a x - \log_a y \qquad (x > 0$ and $y > 0)$

The last two logarithmic laws in the list above are not covered in this section. They are discussed at the end of the chapter and are included for enrichment only.

IMPORTANT: PROOFS ARE NOT REQUIRED FOR EXAMS

Logarithmic law:

$$\log_a x^b = b \log_a x \qquad (x > 0)$$

$$\begin{aligned} \text{Let } \log_a x &= m \ldots (1) \qquad (x > 0) \\ \therefore x &= a^m \\ \therefore (x)^b &= (a^m)^b \\ \therefore x^b &= a^{bm} \\ \text{Change to logarithmic form: } \log_a \left(x^b\right) &= bm \\ \text{And subst : } \quad m &= \log_a x \\ \therefore \log_a x^b &= b \log_a x \end{aligned}$$

In words: the logarithm of a number which is raised to a power is equal to the value of the power multiplied by the logarithm of the number.

See video: 28CC at www.everythingmaths.co.za

Worked example 12: Applying the logarithmic law $\log_a x^b = b \log_a x$

QUESTION

Determine the value of $\log_3 27^4$.

SOLUTION

Step 1: Use the logarithmic law to simplify the expression

$$\begin{aligned}\log_3 27^4 &= 4\log_3 27\\ &= 4\log_3 3^3\\ &= (4\times 3)\log_3 3\\ &= 12(1)\\ &= 12\end{aligned}$$

Step 2: Write the final answer

$\log_3 27^4 = 12$

Special case:

$$\log_a \sqrt[b]{x} = \frac{\log_a x}{b} \qquad (x > 0 \text{ and } b > 0)$$

The following is a special case of the logarithmic law $\log_a x^b = b\log_a x$:

$$\begin{aligned}\log_a \sqrt[b]{x} &= \log_a x^{\frac{1}{b}}\\ &= \frac{1}{b}\log_a x\\ &= \frac{\log_a x}{b}\end{aligned}$$

Exercise 2 – 7: Applying the logarithmic law: $\log_a x^b = b\log_a x$

Simplify the following:

1. $\log_8 10^{10}$
2. $\log_{16} x^y$
3. $\log_3 \sqrt{5}$
4. $\log_z y^z$
5. $\log_y \sqrt[x]{y}$
6. $\log_p p^q$
7. $\log_2 \sqrt[4]{8}$
8. $\log_5 \frac{1}{5}$
9. $\log_2 8^5$
10. $\log_4 16 \times \log_3 81$
11. $(\log_5 25)^2$
12. $\log_2 0{,}125$

Check answers online with the exercise code below or click on 'show me the answer'.

1. 28CD 2. 28CF 3. 28CG 4. 28CH 5. 28CJ 6. 28CK
7. 28CM 8. 28CN 9. 28CP 10. 28CQ 11. 28CR 12. 28CS

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Logarithmic law: $\log_a x = \frac{\log_b x}{\log_b a}$ $\quad (b > 0$ and $b \neq 1)$

It is often necessary or convenient to convert a logarithm from one base to another base. This is referred to as a **change of base**.

$$\begin{aligned} \text{Let} \quad \log_a x &= m \\ \therefore x &= a^m \\ \text{Consider the fraction:} \quad & \frac{\log_b x}{\log_b a} \\ \text{Substitute } x = a^m: \quad \frac{\log_b x}{\log_b a} &= \frac{\log_b a^m}{\log_b a} \\ &= m\left(\frac{\log_b a}{\log_b a}\right) \\ &= m(1) \\ \therefore \frac{\log_b x}{\log_b a} &= \log_a x \end{aligned}$$

Special applications:

$$\begin{aligned} (1) \qquad \log_a x &= \frac{\log_x x}{\log_x a} \\ \therefore \log_a x &= \frac{1}{\log_x a} \end{aligned}$$

$$\begin{aligned} (2) \qquad \log_a \frac{1}{x} &= \log_a x^{-1} \\ \therefore \log_a \frac{1}{x} &= -\log_a x \end{aligned}$$

See video: 28CT at www.everythingmaths.co.za

Worked example 13: Applying the logarithmic law $\log_a x = \frac{\log_b x}{\log_b a}$

QUESTION

Show: $\log_2 8 = \frac{\log 8}{\log 2}$

SOLUTION

Step 1: Simplify the right-hand side of the equation

$$\begin{aligned} \text{RHS} &= \frac{\log 8}{\log 2} \\ &= \frac{\log 2^3}{\log 2} \\ &= 3\left(\frac{\log 2}{\log 2}\right) \\ &= 3(1) \\ &= 3 \end{aligned}$$

Step 2: Simplify the left-hand side of the equation

$$\begin{aligned} \text{LHS} &= \log_2 8 \\ &= \log_2 2^3 \\ &= 3\log_2 2 \\ &= 3(1) \\ &= 3 \end{aligned}$$

Step 3: Write the final answer

We have shown that $\log_2 8 = \frac{\log 8}{\log 2} = 3$.

Worked example 14: Applying the logarithmic law $\log_a x = \frac{\log_b x}{\log_b a}$

QUESTION

If $a = \log 2$ and $b = \log 3$, express the following in terms of a and b:

1. $\log_3 2$
2. $\log_2 \frac{10}{3}$

SOLUTION

Step 1: Use a change of base to simplify the expressions

$$\begin{aligned} \log_3 2 &= \frac{\log 2}{\log 3} \\ &= \frac{a}{b} \end{aligned}$$

$$\begin{aligned} \log_2 \frac{10}{3} &= \frac{\log \frac{10}{3}}{\log 2} \\ &= \frac{\log 10 - \log 3}{\log 2} \\ &= \frac{1-b}{a} \end{aligned}$$

Exercise 2 – 8: Applying the logarithmic law: $\log_a x = \frac{\log_b x}{\log_b a}$

1. Convert the following:

   a) $\log_2 4$ to base 8

   b) $\log_{10} 14$ to base 2

   c) $\log 4\frac{1}{2}$ to base 2

   d) $\log_2 8$ to base 8

   e) $\log_y x$ to base x

   f) $\log_{10} 2x$ to base 2

2. Simplify the following using a change of base:

   a) $\log_2 10 \times \log_{10} 2$

   b) $\log_5 100$

3. If $\log 3 = 0{,}477$ and $\log 2 = 0{,}301$, determine (correct to 2 decimal places):

   a) $\log_2 3$

   b) $\log_3 2000$

4. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 28CV 1b. 28CW 1c. 28CX 1d. 28CY 1e. 28CZ 1f. 28D2
2a. 28D3 2b. 28D4 3a. 28D5 3b. 28D6

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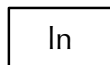
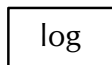
Logarithms using a calculator EMCFM

Calculating a logarithmic value

There are many different types and models of scientific calculators. It is very important to be familiar with your own calculator and the different function buttons. Some calculators only have two buttons for logarithms: one for calculating the common logarithm (base is equal to 10) and another for calculating the natural log (base is equal to e). Newer models will have a third button which allows the user to calculate the logarithm of a number to a certain base.

log	ln	log□□

Worked example 15: Using a calculator: logarithm function

QUESTION

Use a calculator to determine the following values (correct to 3 decimal places):

1. $\log 9$
2. $\log 0{,}3$
3. $\log \frac{3}{4}$
4. $\log(-2)$

SOLUTION

Step 1: Use the common logarithm function on your calculator

Make sure that you are familiar with the "LOG" function on your calculator. Notice that the base for each of the logarithms given above is 10.

1. log 9 = 0,954...
2. log 0 , 3 = −0,522...
3. log (3 ÷ 4) = −0,124...
4. log (− 2) = *ERROR*

Step 2: Write the final answer

1. $\log 9 = 0{,}954$
2. $\log 0{,}3 = -0{,}523$
3. $\log \frac{3}{4} = -0{,}125$
4. $\log(-2) =$ undefined

Worked example 16: Using a calculator: inverse logarithm function

QUESTION

Use a calculator to determine the following values (correct to 3 decimal places):

1. $\log x = 1{,}7$
2. $\log t = \frac{2}{7}$
3. $\log y = -3$

SOLUTION

Step 1: Use the second function and common logarithm function on your calculator

For each of the logarithms given above we need to calculate the inverse of the logarithm (sometimes called the antilog). Make sure that you are familiar with the "2nd F" button on your calculator.

Notice that by pressing the "2nd F" button and then the "LOG" button, we are using the "10^x" function on the calculator, which is correct since exponentials are the inverse of logarithms.

1. 2ndF log 1 , 7 = 50,118...
2. 2ndF log (2 ÷ 7) = 1,930...
3. 2ndF log (− 3) = 0,001

Step 2: Write the final answer

1. $x = 50{,}119$
2. $t = 1{,}930$
3. $y = 0{,}001$

Worked example 17: Using a calculator: change of base

QUESTION

Use a calculator to find $\log_2 5$ correct to two decimal places.

SOLUTION

Step 1:

$$\log_2 5 = \frac{\log 5}{\log 2}$$

Step 2: Use a change of base to convert given logarithm to base 10

$$\log_2 5 = \frac{\log 5}{\log 2}$$

Step 3: Use the common logarithm function on your calculator

log	5	÷	log	2	=	$2,321\ldots$

Step 4: Write the final answer

$$\log_2 5 = 2{,}32$$

Important:

- Do not write down an intermediate step when doing this type of calculation:

$$\begin{aligned}\log_2 5 &= \frac{\log 5}{\log 2}\\ &= \frac{0{,}7}{0{,}3} \quad \text{(this step can cause rounding off errors)}\\ &= 2{,}33\end{aligned}$$

  Perform the calculation in one step on your calculator:

$$\begin{aligned}\log_2 5 &= \frac{\log 5}{\log 2}\\ &= 2{,}32\end{aligned}$$

- Do not round off before the final answer as this can affect the accuracy of the answer.
- Be sure that you determine the correct sequence and order of operations when using a calculator.

Exercise 2 – 9: Logarithms using a calculator

1. Calculate the following (correct to three decimal places):

a) $\log 3$
b) $\log 30$
c) $\log 300$
d) $\log 0.66$
e) $\log \frac{1}{4}$
f) $\log 852$
g) $\log(-6)$
h) $\log_3 4$
i) $\log 0{,}01$
j) $\log_2 15$
k) $\log_4 10$
l) $\log_{\frac{1}{2}} 6$

2. Use a calculator to determine the value of x (correct to two decimal places). Check your answer by changing to exponential form.

a) $\log x = 0{,}6$
b) $\log x = -2$
c) $\log x = 1{,}8$
d) $\log x = 5$
e) $\log x = -0{,}5$
f) $\log x = 0{,}076$
g) $\log x = \frac{2}{5}$
h) $\log x = -\frac{6}{5}$
i) $\log_2 x = 0{,}25$
j) $\log_5 x = -0{,}1$
k) $\log_{\frac{1}{4}} x = 2$
l) $\log_7 x = 0{,}3$

3. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 28D7	1b. 28D8	1c. 28D9	1d. 28DB	1e. 28DC	1f. 28DD
1g. 28DF	1h. 28DG	1i. 28DH	1j. 28DJ	1k. 28DK	1l. 28DM
2a. 28DN	2b. 28DP	2c. 28DQ	2d. 28DR	2e. 28DS	2f. 28DT
2g. 28DV	2h. 28DW	2i. 28DX	2j. 28DY	2k. 28DZ	2l. 28F2

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Exponential and logarithmic graphs EMCFN

Worked example 18: Graphs of the inverse of $y = b^x$

QUESTION

On the same system of axes, draw the graphs of $f(x) = 10^x$ and its inverse $f^{-1}(x) = \log x$. Investigate the properties of f and f^{-1}.

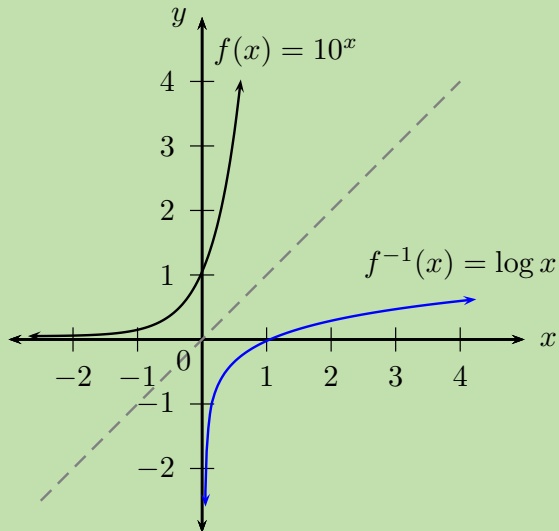
SOLUTION

Step 1: Determine the properties of $f(x)$

- Function: $y = 10^x$
- Shape: increasing graph

- Intercept(s): $(0; 1)$
- Asymptote(s): horizontal asymptote at x-axis, line $y = 0$
- Domain: $\{x : x \in \mathbb{R}\}$
- Range: $\{y : y > 0, y \in \mathbb{R}\}$

Step 2: Draw the graphs

The graph of the inverse f^{-1} is the reflection of f about the line $y = x$.

Step 3: Determine the properties of $f^{-1}(x)$

- Function: $y = \log x$
- Shape: increasing graph
- Intercept(s): $(1; 0)$
- Asymptote(s): vertical asymptote at y-axis, line $x = 0$
- Domain: $\{x : x > 0, x \in \mathbb{R}\}$
- Range: $\{y : y \in \mathbb{R}\}$

Notice that the inverse is a function: $f^{-1}(x) = \log x$ is a one-to-one function since every input value is associated with only one output value.

The exponential function and the logarithmic function are inverses of each other:

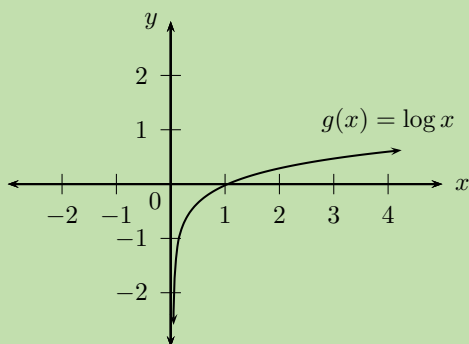
- the domain of the function is equal to the range of the inverse
- the range of the function is equal to the domain of the inverse
- the y-intercept of the function is equal to the x-intercept of the inverse
- the x-intercept of the function is equal to the y-intercept of the inverse
- the asymptote for the function is $y = 0$ and the asymptote for the inverse is $x = 0$
- the graphs are reflected about the line $y = x$

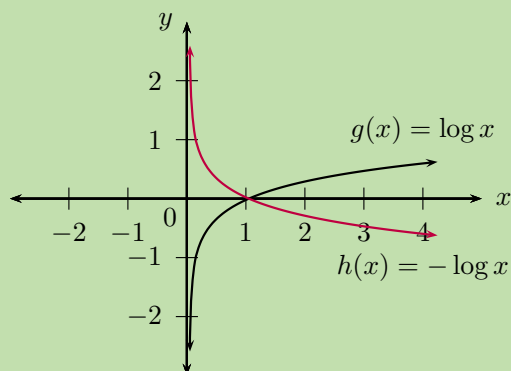
Worked example 19: Graphs of $y = \log_b x$

QUESTION

1. Draw a sketch of $g(x) = \log_{10} x$.
2. Reflect the graph of g about the x-axis to give the graph h.
3. Investigate the properties of h.
4. Use g and h to suggest a general conclusion.

SOLUTION

Step 1: Sketch the graph of $g(x) = \log_{10} x$

Step 2: Reflect g about the x-axis

An easy method for reflecting a graph about a certain line is to imagine folding the Cartesian plane along that line and the reflected graph is pressed onto the plane.

Step 3: Investigate the properties of h

- Function: passes the vertical line test
- Shape: decreasing graph
- Intercept(s): $(1; 0)$
- Asymptote(s): vertical asymptote at y-axis, line $x = 0$
- Domain: $\{x : x > 0, x \in \mathbb{R}\}$
- Range: $\{y : y \in \mathbb{R}\}$

Since $h(x)$ is symmetrical to $g(x)$ about the x-axis, this means that every y-value of g corresponds to a y-value of the opposite sign for h.

$$\begin{aligned}
\text{Given } g(x) &= \log_{10} x \\
\therefore h(x) &= -\log_{10} x \\
\text{Let } y &= -\log_{10} x \\
-y &= \log_{10} x \\
\therefore 10^{-y} &= x \\
\left(\frac{1}{10}\right)^{y} &= x \\
\therefore y &= \log_{\frac{1}{10}} x \\
\therefore h(x) &= -\log_{10} x = \log_{\frac{1}{10}} x
\end{aligned}$$

Step 4: General conclusion

From this example of g and h we see that:

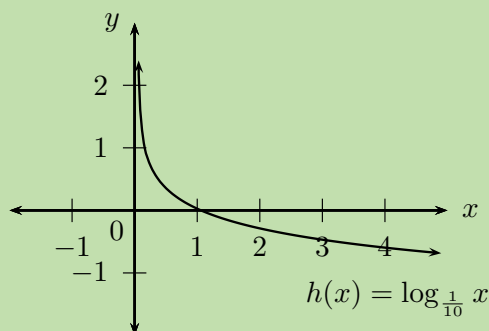
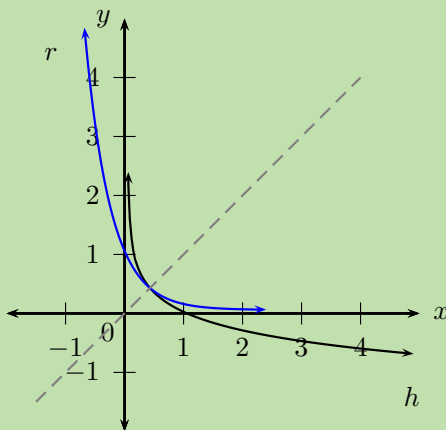
$$-\log_{m} p = \log_{\frac{1}{m}} p$$

Worked example 20: Graph of $y = \log_{b} x$

QUESTION

1. Draw a sketch of $h(x) = \log_{\frac{1}{10}} x$.
2. Draw the graph of $r(x)$, the reflection of h about the line $y = x$.
3. Investigate the properties of r.
4. Write down the new equation if h is shifted 1 unit upwards and 2 units to the right.

SOLUTION

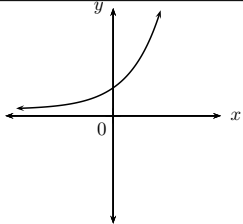
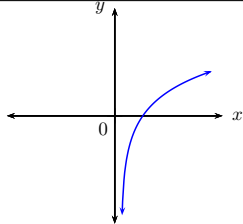
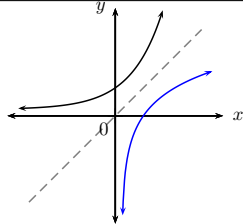
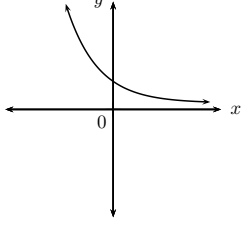
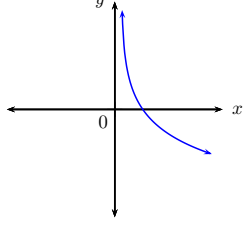
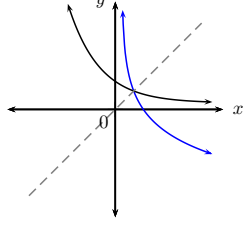
Step 1: Sketch the graph of $h(x) = \log_{\frac{1}{10}} x$

Step 2: Reflect h about the line $y = x$

Step 3: Investigate the properties of r

- Function: passes the vertical line test
- Shape: decreasing graph
- Intercept(s): $(0; 1)$
- Asymptote(s): horizontal asymptote at x-axis, line $y = 0$
- Domain: $\{x : x \in \mathbb{R}\}$
- Range: $\{y : y > 0, y \in \mathbb{R}\}$

Since $h(x)$ is symmetrical to $r(x)$ about the line $y = x$, this means that r is the inverse of h.

$$\begin{aligned} h(x) &= \log_{\frac{1}{10}} x \\ \text{Let } y &= \log_{\frac{1}{10}} x \\ \text{Inverse: } x &= \log_{\frac{1}{10}} y \\ \therefore \left(\frac{1}{10}\right)^x &= y \\ 10^{-x} &= y \end{aligned}$$

$$\therefore r(x) = h^{-1}(x) = 10^{-x}$$

Therefore, $r(x)$ is an exponential function of the form $y = b^x$ with $0 < b < 1$. In words, the base b is a positive fraction with a value between 0 and 1.

Step 4: Vertical and horizontal shifts

If h is shifted 1 unit upwards and 2 units to the right, then the new equation will be:

$$y = \log_{\frac{1}{10}} (x - 2) + 1$$

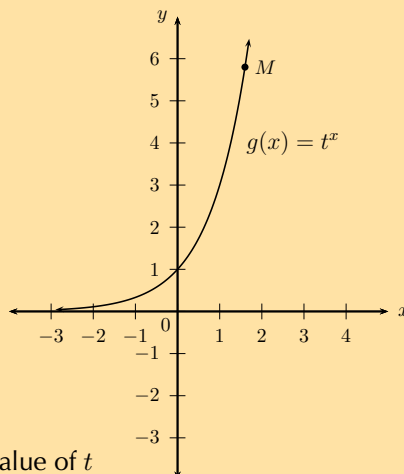
The vertical asymptote is $x = 2$ and the horizontal asymptote is $y = 1$.

Summary of graphs: $y = b^x$ **and** $y = \log_b x$

	Exponential function	Logarithmic function	Axis of symmetry
	$y = b^x$	$y = \log_b x$	$y = x$
$b > 1$			
$0 < b < 1$			
Axis of symmetry	y-axis, $x = 0$	x-axis, $y = 0$	

See video: 28F3 at www.everythingmaths.co.za

Exercise 2 – 10: Graphs and inverses of $y = \log_b x$

1. Given $f(x) = \left(\frac{1}{5}\right)^x$.
   a) Sketch the graphs of f and f^{-1} on the same system of axes. Label both graphs clearly.
   b) State the intercept(s) for each graph.
   c) Label P, the point of intersection of f and f^{-1}.
   d) State the domain, range and asymptote(s) of each function.
2. Given $g(x) = t^x$ with $M\left(1\frac{3}{5}; 5\frac{4}{5}\right)$ a point on the graph of g.

   a) Determine the value of t
   b) Find the inverse of g.

c) Use symmetry about the line $y = x$ to sketch the graphs of g and g^{-1} on the same system of axes.

d) Point N lies on the graph of g^{-1} and is symmetrical to point M about the line $y = x$. Determine the coordinates of N.

3. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 28F4 2. 28F5

Applications of logarithms EMCFP

Logarithms have many different applications:

- seismologists use logarithms to calculate the magnitude of earthquakes
- financial institutions make use of logarithms to calculate the length of loan repayments
- scientists use logarithms to determine the rate of radioactive decay
- biologists use logarithms to calculate population growth rates
- scientists use logarithms to determine pH levels

$$\text{pH} = -\log_{10}[H^+]$$

See video: 28F6 at www.everythingmaths.co.za

Worked example 21: Population growth

QUESTION

The population of a city grows by 5% every two years. How long will it take for the city's population to triple in size?

SOLUTION

Step 1: Write down a suitable formula and the known values

$$A = P(1+i)^n$$

- Let $P = x$
- The population triples in size, so $A = 3x$
- Growth rate $i = \frac{5}{100}$
- Growth rate is given for a 2 year period, so we use $\frac{n}{2}$

Step 2: Substitute known values and solve for n

$$3x = x\left(1+\frac{5}{100}\right)^{\frac{n}{2}}$$
$$3 = (1,05)^{\frac{n}{2}}$$

Step 3: Method 1: take the logarithm of both sides of the equation

$$\begin{aligned} \log 3 &= \log (1,05)^{\frac{n}{2}} \\ \log 3 &= \frac{n}{2} \times \log 1,05 \\ 2 \times \frac{\log 3}{\log 1,05} &= n \\ 45,034\ldots &= n \end{aligned}$$

Step 4: Method 2: change from exponential form to logarithmic form

$$\begin{aligned} \frac{n}{2} &= \log_{1,05} 3 \\ &= \frac{\log 3}{\log 1{,}05} \\ n &= 2 \times \frac{\log 3}{\log 1,05} \\ n &= 45,034\ldots \end{aligned}$$

Step 5: Write the final answer

It will take approximately 45 years for the city's population to triple in size.

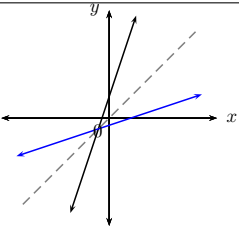
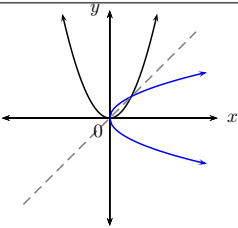
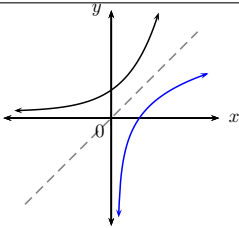
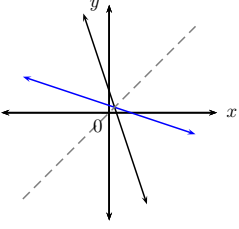
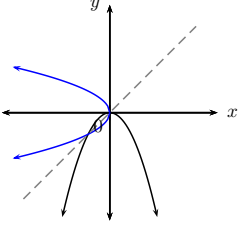
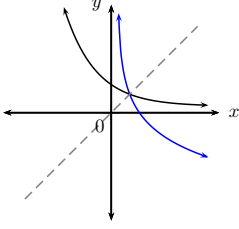
Exercise 2 – 11: Applications of logarithms

1. The population of Upington grows 6% every 3 years. How long will it take to triple in size?
   Give your answer in years and round to the nearest integer.
2. An ant population of 36 ants doubles every month.
   a) Determine a formula that describes the growth of the population.
   b) Calculate how long it will take for the ant population to reach a quarter of a million ants.
3. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 28F7 2. 28F8

2.7 Summary EMCFQ

See presentation: 28F9 at www.everythingmaths.co.za

- Function: a rule which uniquely associates elements of one set A with the elements of another set B; each element in set A maps to only one element in set B.
- Functions can be one-to-one relations or many-to-one relations. A many-to-one relation associates two or more values of the independent (input) variable with a single value of the dependent (output) variable.
- Vertical line test: if it is possible to draw any vertical line which crosses the graph of the relation more than once, then the relation is not a function.
- Given the invertible function $f(x)$, we determine the inverse $f^{-1}(x)$ by:
  - replacing every x with y and y with x;
  - making y the subject of the equation;
  - expressing the new equation in function notation.

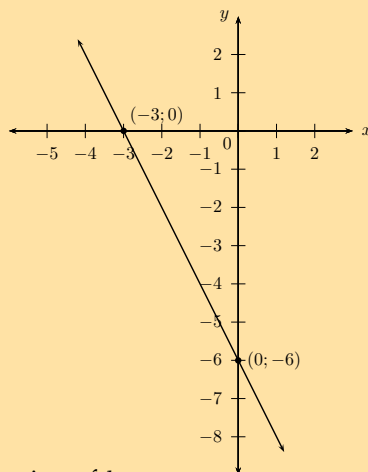
  If we represent the function f and the inverse function f^{-1} graphically, the two graphs are reflected about the line $y = x$.
- The domain of the function is equal to the range of the inverse. The range of the function is equal to the domain of the inverse.
- The inverse function of a straight line is also a straight line. Vertical and horizontal lines are exceptions.
- The inverse of a parabola is not a function. However, we can limit the domain of the parabola so that the inverse of the parabola is a function.
- The inverse of the exponential function $f(x) = b^x, (b > 0, b \neq 1)$ is the logarithmic function $f^{-1}(x) = \log_b x$.
- The "common logarithm" has a base 10 and can be written as $\log_{10} x = \log x$. The $\log$ symbol written without a base means $\log$ base 10.

Logarithmic laws:

- $\log_a x^b = b \log_a x \qquad (x > 0)$
- $\log_a x = \frac{\log_b x}{\log_b a} \qquad (b > 0$ and $b \neq 1)$
- $\log_a xy = \log_a x + \log_a y \qquad (x > 0$ and $y > 0)$
- $\log_a \frac{x}{y} = \log_a x - \log_a y \qquad (x > 0$ and $y > 0)$

	Straight line function	Quadratic function	Exponential function
Formula	$y = ax + q$	$y = ax^2$	$y = b^x$
Inverse	$y = \frac{x}{a} - \frac{q}{a}$	$y = \pm\sqrt{\frac{x}{a}}$	$y = \log_b x$
Inverse a function?	yes	no	yes
Graphs			

Exercise 2 – 12: End of chapter exercises

1. Given the straight line h with intercepts $(-3; 0)$ and $(0; -6)$.

   a) Determine the equation of h.
   b) Find h^{-1}.
   c) Draw both graphs on the same system of axes.
   d) Calculate the coordinates of S, the point of intersection of h and h^{-1}.
   e) State the property regarding the point of intersection that will always be true for a function and its inverse.

2. The inverse of a function is $f^{-1}(x) = 2x + 4$.
   a) Determine f.
   b) Draw f and f^{-1} on the same set of axes. Label each graph clearly.
   c) Is f^{-1} an increasing or decreasing function? Explain your answer.

3. $f(x) = 2x^2$.

   a) Draw the graph of f and state its domain and range.
   b) Determine the inverse and state its domain and range.

4. Given the function $f(x) = \left(\frac{1}{4}\right)^x$.

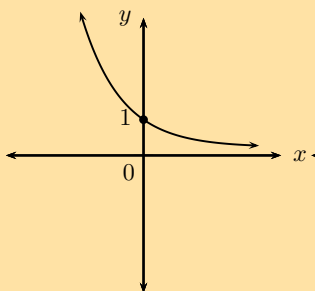
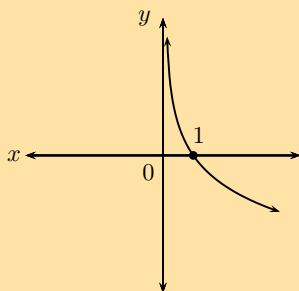
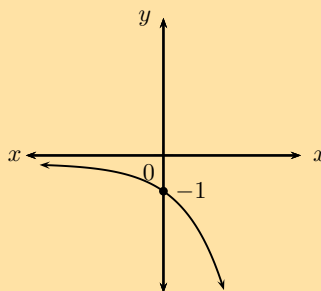
   a) Sketch the graphs of f and f^{-1} on the same system of axes.
   b) Determine if the point $\left(-\frac{1}{2}; 2\right)$ lies on the graph of f.
   c) Write f^{-1} in the form $y = \ldots$
   d) If the graphs of f and f^{-1} intersect at $\left(\frac{1}{2}; P\right)$, determine the value of P.
   e) Give the equation of the new graph, G, if the graph of f^{-1} is shifted 2 units to the left.
   f) Give the asymptote(s) of G.

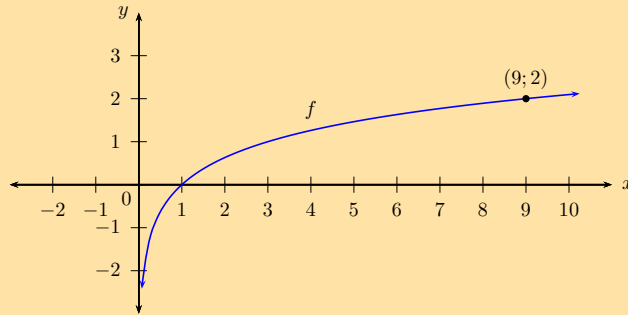
5. Consider the function $h(x) = 3^x$.

   a) Write down the inverse in the form $h^{-1}(x) = \ldots$
   b) State the domain and range of h^{-1}.
   c) Sketch the graphs of h and h^{-1} on the same system of axes, label all intercepts.
   d) For which values of x will $h^{-1}(x) < 0$?

6. Consider the functions $f(x) = 2^x$ and $g(x) = x^2$.

   a) Sketch the graphs of f and g on the same system of axes.
   b) Determine whether or not f and g intersect at a point where $x = -1$.
   c) How many solutions does the equation $2^x = x^2$ have?

7. Below are three graphs and six equations. Write down the equation that best matches each of the graphs.

   a) $y = \log_3 x$
   b) $y = -\log_3 x$
   c) $y = \log_{\frac{1}{3}} x$
   d) $y = 3^x$
   e) $y = 3^{-x}$
   f) $y = -3^x$

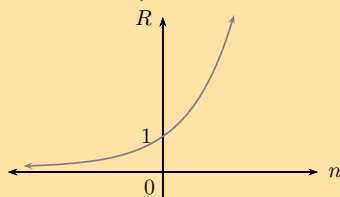
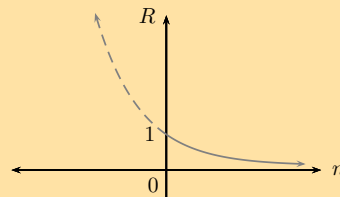
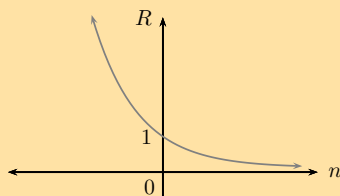
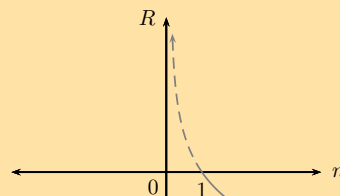
8. Given the graph of the function $f: \ y = \log_b x$ passing through the point $(9;2)$.

a) Show that $b = 3$.

b) Determine the value of a if $(a;-1)$ lies on f.

c) Write down the new equation if f is shifted 2 units upwards.

d) Write down the new equation if f is shifted 1 units to the right.

9. a) If the rhino population in South Africa starts to decrease at a rate of 7% per annum, determine how long it will take for the current rhino population to halve in size? Give your answer to the nearest integer.

b) Which of the following graphs best illustrates the rhino population's decline? Motivate your answer.

Important note: the graphs above have been drawn as a continuous curve to show a trend. Rhino population numbers are discrete values and should be plotted points.

10. At 8 a.m. a local celebrity tweets about his new music album to 100 of his followers. Five minutes later, each of his followers retweet his message to two of their friends. Five minutes after that, each friend retweets the message to another two friends. Assume this process continues.

a) Determine a formula that describes this retweeting process.

b) Calculate how many retweets of the celebrity's message are sent an hour after his original tweet.

1 hour $= 60$ minutes $= 12 \times 5$, therefore $n = 12$.

c) How long will it take for the total number of retweets to exceed 200 million?

11. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 28FB	1b. 28FC	1c. 28FD	1d. 28FF	1e. 28FG	2. 28FH
3a. 28FJ	3b. 28FK	4. 28FM	5. 28FN	6. 28FP	7. 28FQ
8. 28FR	9. 28FS	10. 28FT			

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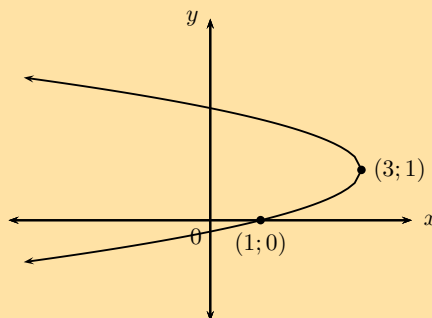
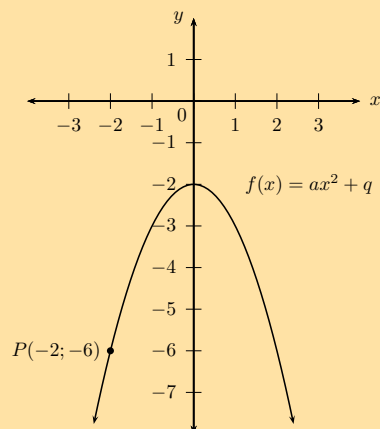
Exercise 2 – 13: Inverses (ENRICHMENT ONLY)

1. a) Given: $g(x) = -1 + \sqrt{x}$, find the inverse of $g(x)$ in the form $g^{-1}(x) = ...$
   b) Draw the graph of g^{-1}.
   c) Use symmetry to draw the graph of g on the same set of axes.
   d) Is g^{-1} a function?
   e) Give the domain and range of g^{-1}.
2. The graph of the inverse of f is shown below:

   a) Find the equation of f, given that f is a parabola of the form $y = (x+p)^2 + q$.
   b) Will f have a maximum or a minimum value?
   c) State the domain, range and axis of symmetry of f.
3. Given: $k(x) = 2x^2 + 1$
   a) If $(q; 3)$ lies on k, determine the value(s) of q.
   b) Sketch the graph of k, label the point(s) $(q; 3)$ on the graph.
   c) Find the equation of the inverse of k in the form $y = \ldots$
   d) Sketch k and $y = \sqrt{\frac{x-1}{2}}$ on the same system of axes.
   e) Determine the coordinates of the point on the graph of the inverse that is symmetrical to $(q; 3)$ about the line $y = x$.

4. The sketch shows the graph of a parabola $f(x) = ax^2 + q$ passing through the point $P(-2;-6)$.

   a) Determine the equation of f.
   b) Determine and investigate the inverse.
   c) Sketch the inverse and discuss the characteristics of the graph.

5. Given the function $H : y = x^2 - 9$.
   a) Determine the algebraic formula for the inverse of H.
   b) Draw graphs of H and its inverse on the same system of axes. Indicate intercepts and turning points.
   c) Is the inverse a function? Give reasons.
   d) Show algebraically and graphically the effect of restricting the domain of H to $\{x : x \leq 0\}$.

6. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 28FV 2. 28FW 3. 28FX 4a. 28FY 4b. 28FZ 4c. 28G2
5. 28G3

2.8 Enrichment: more on logarithms EMCFR

NOTE: THIS SECTION IS NOT PART OF THE CURRICULUM

Laws of logarithms EMCFS

Logarithmic law:

$$\log_a xy = \log_a x + \log_a y \qquad (x > 0 \text{ and } y > 0)$$

$$\begin{aligned} \text{Let } \log_a(x) = m &\implies x = a^m \ldots (1) \qquad (x > 0) \\ \text{and } \log_a(y) = n &\implies x = a^n \ldots (2) \qquad (y > 0) \end{aligned}$$

$$\begin{aligned} \text{Then } (1) \times (2): \quad & x \times y = a^m \times a^n \\ & \therefore xy = a^{m+n} \end{aligned}$$

Now we change from the exponential form back to logarithmic form:

$$\begin{aligned} \log_a xy &= m + n \\ \text{But } m = \log_a(x) &\text{ and } n = \log_a(y) \\ \therefore \log_a xy &= \log_a(x) + \log_a(y) \end{aligned}$$

In words: the logarithm of a product is equal to the sum of the logarithms of the factors.

Worked example 22: Applying the logarithmic law $\log_a xy = \log_a x + \log_a y$

QUESTION

Simplify: $\log 5 + \log 2 - \log 30$

SOLUTION

Step 1: Use the logarithmic law to simplify the expression

We combine the first two terms since the product of 5 and 2 is equal to 10, which is always useful when simplifying logarithms.

$$\begin{aligned} \log 5 + \log 2 - \log 30 &= (\log 5 + \log 2) - \log 30 \\ &= \log(5 \times 2) - \log 30 \\ &= \log 10 - \log 30 \\ &= 1 - \log 30 \end{aligned}$$

We expand the last term to simplify the expression further:

$$
\begin{aligned}
&= 1 - \log(3 \times 10) \\
&= 1 - (\log 3 + \log 10) \\
&= 1 - (\log 3 + 1) \\
&= 1 - \log 3 - 1 \\
&= -\log 3
\end{aligned}
$$

Step 2: Write the final answer

$\log 5 + \log 2 - \log 30 = -\log 3$

Exercise 2 – 14: Applying logarithmic law: $\log_a xy = \log_a(x) + \log_a(y)$

1. Simplify the following, if possible:

   a) $\log_8(10 \times 10)$
   b) $\log_2 14$
   c) $\log_2(8 \times 5)$
   d) $\log_{16}(x + y)$
   e) $\log_2 2xy$
   f) $\log(5 + 2)$

2. Write the following as a single term, if possible:

   a) $\log 15 + \log 2$
   b) $\log 1 + \log 5 + \log \frac{1}{5}$
   c) $1 + \log_3 4$
   d) $(\log x)(\log y) + \log x$
   e) $\log 7 \times \log 2$
   f) $\log_2 7 + \log_3 2$
   g) $\log_a p + \log_a q$
   h) $\log_a p \times \log_a q$

3. Simplify the following:

   a) $\log x + \log y + \log z$
   b) $\log ab + \log bc + \log cd$
   c) $\log 125 + \log 2 + \log 8$
   d) $\log_4 \frac{3}{8} + \log_4 \frac{10}{3} + \log_4 \frac{16}{5}$

4. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 28G4	1b. 28G5	1c. 28G6	1d. 28G7	1e. 28G8	1f. 28G9
2a. 28GB	2b. 28GC	2c. 28GD	2d. 28GF	2e. 28GG	2f. 28GH
2g. 28GJ	2h. 28GK	3a. 28GM	3b. 28GN	3c. 28GP	3d. 28GQ

Logarithmic law:

$$\log_a \frac{x}{y} = \log_a x - \log_a y \qquad (x > 0 \text{ and } y > 0)$$

$$\text{Let } \log_a (x) = m \implies x = a^m \ldots (1) \qquad (x > 0)$$
$$\text{and } \log_a (y) = n \implies y = a^n \ldots (2) \qquad (y > 0)$$

$$\text{Then } (1) \div (2): \quad \frac{x}{y} = \frac{a^m}{a^n}$$
$$\therefore \frac{x}{y} = a^{m-n}$$

Now we change from the exponential form back to logarithmic form:

$$\log_a \frac{x}{y} = m - n$$
$$\text{But } m = \log_a (x) \text{ and } n = \log_a (y)$$
$$\therefore \log_a \frac{x}{y} = \log_a (x) - \log_a (y)$$

In words: the logarithm of a quotient is equal to the difference of the logarithms of the numerator and the denominator.

Worked example 23: Applying the logarithmic law $\log_a \frac{x}{y} = \log_a x - \log_a y$

QUESTION

Simplify: $\log 40 - \log 4 + \log_5 \frac{8}{5}$

SOLUTION

Step 1: Use the logarithmic law to simplify the expression

We combine the first two terms since both terms have the same base and the quotient of 40 and 4 is equal to 10:

$$\begin{aligned}\log 40 - \log 4 + \log_5 \frac{8}{5} &= (\log 40 - \log 4) + \log_5 \frac{8}{5}\\ &= \left(\log \frac{40}{4}\right) + \log_5 \frac{8}{5}\\ &= \log 10 + \log_5 \frac{8}{5}\\ &= 1 + \log_5 \frac{8}{5}\end{aligned}$$

We expand the last term to simplify the expression further:

$$\begin{aligned}&= 1 + (\log_5 8 - \log_5 5)\\ &= 1 + \log_5 8 - 1\\ &= \log_5 8\end{aligned}$$

Step 2: Write the final answer

$\log 40 - \log 4 + \log_5 \frac{8}{5} = \log_5 8$

Exercise 2 – 15: Applying logarithmic law: $\log_a \frac{x}{y} = \log_a x - \log_a y$

1. Expand and simplify the following:

   a) $\log \frac{100}{3}$
   b) $\log_2 7\frac{1}{2}$
   c) $\log_{16} \frac{x}{y}$
   d) $\log_{16} (x - y)$
   e) $\log_5 \frac{5}{8}$
   f) $\log_x \frac{y}{r}$

2. Write the following as a single term:

   a) $\log 10 - \log 50$
   b) $\log_3 36 - \log_3 4$
   c) $\log_a p - \log_a q$
   d) $\log_a (p - q)$
   e) $\log 15 - \log_2 5$
   f) $\log 15 - \log 5$

3. Simplify the following:

   a) $\log 450 - \log 9 - \log 5$
   b) $\log \frac{4}{5} - \log \frac{3}{25} - \log \frac{1}{15}$

4. Vini and Dirk complete their mathematics homework and check each other's answers. Compare the two methods shown below and decide if they are correct or incorrect:

   Question:

   Simplify the following:

   $$\log m - \log n - \log p - \log q$$

   Vini's answer:

   $$\begin{aligned}\log m - \log n - \log p - \log q &= (\log m - \log n) - \log p - \log q \\ &= \left(\log \frac{m}{n} - \log p\right) - \log q \\ &= \log \left(\frac{m}{n} \times \frac{1}{p}\right) - \log q \\ &= \log \frac{m}{np} - \log q \\ &= \log \frac{m}{np} \times \frac{1}{q} \\ &= \log \frac{m}{npq}\end{aligned}$$

   Dirk's answer:

   $$\begin{aligned}\log m - \log n - \log p - \log q &= \log m - (\log n + \log p + \log q) \\ &= \log m - \log (n \times p \times q) \\ &= \log m - \log (npq) \\ &= \log \frac{m}{npq}\end{aligned}$$

5. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 28GR 1b. 28GS 1c. 28GT 1d. 28GV 1e. 28GW 1f. 28GX
2a. 28GY 2b. 28GZ 2c. 28H2 2d. 28H3 2e. 28H4 2f. 28H5
3a. 28H6 3b. 28H7 4. 28H8

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Useful summary:

1. $\log 1 = 0$
2. $\log 10 = 1$
3. $\log 100 = 2$
4. $\log 1000 = 3$
5. $\log \frac{1}{10} = -1$
6. $\log 0{,}1 = -1$
7. $\log 0{,}01 = -2$
8. $\log 0{,}001 = -3$

Simplification of logarithms EMCFT

Worked example 24: Simplification of logarithms

QUESTION

Simplify (without a calculator): $3 \log 3 + \log 125$

SOLUTION

Step 1: Apply the appropriate logarithmic laws to simplify the expression

$$\begin{aligned} 3 \log 3 + \log 125 &= 3 \log 3 + \log 5^3 \\ &= 3 \log 3 + 3 \log 5 \\ &= 3 (\log 3 + \log 5) \\ &= 3 \log (3 \times 5) \\ &= 3 \log 15 \end{aligned}$$

Step 2: Write the final answer

We cannot simplify any further, therefore $3 \log 3 + \log 125 = 3 \log 15$.

Important: all the algebraic manipulation techniques $(\times, \div, +, -$, factorisation etc.) also apply for logarithmic expressions. Always be aware of the number of terms in an expression as this will help to determine how to simplify.

Exercise 2 – 16: Simplification of logarithms

Simplify the following without using a calculator:

1. $8^{\frac{2}{3}} + \log_2 32$
2. $2 \log 3 + \log 2 - \log 5$
3. $\log_2 8 - \log 1 + \log_4 \frac{1}{4}$
4. $\log_8 1 - \log_5 \frac{1}{25} + \log_3 9$

Check answers online with the exercise code below or click on 'show me the answer'.

1. 28H9 2. 28HB 3. 28HC 4. 28HD

Solving logarithmic equations EMCFV

Worked example 25: Solving logarithmic equations

QUESTION

Solve for p:

$$18 \log p - 36 = 0$$

SOLUTION

Step 1: Make $\log p$ the subject of the equation

$$\begin{aligned} 18 \log p - 36 &= 0 \\ 18 \log p &= 36 \\ \frac{18 \log p}{18} &= \frac{36}{18} \\ \therefore \log p &= 2 \end{aligned}$$

Step 2: Change from logarithmic form to exponential form

$$\begin{aligned} \log p &= 2 \\ \therefore p &= 10^2 \\ &= 100 \end{aligned}$$

Step 3: Write the final answer

$p = 100$

Worked example 26: Solving logarithmic equations

QUESTION

Solve for n (correct to the nearest integer):

$$(1{,}02)^n = 2$$

SOLUTION

Step 1: Change from exponential form to logarithmic form

$$(1{,}02)^n = 2$$
$$\therefore n = \log_{1{,}02} 2$$

Step 2: Use a change of base to solve for n

$$n = \frac{\log 2}{\log 1{,}02}$$
$$\therefore n = 35{,}00\ldots$$

Step 3: Write the final answer

$n = 35$

Exercise 2 – 17: Solving logarithmic equations

1. Determine the value of a (correct to 2 decimal places):

    a) $\log_3 a - \log 1{,}2 = 0$
    b) $\log_2 (a - 1) = 1{,}5$
    c) $\log_2 a - 1 = 1{,}5$
    d) $3^a = 2{,}2$
    e) $2^{(a+1)} = 0{,}7$
    f) $(1{,}03)^{\frac{a}{2}} = 2{,}65$
    g) $(9)^{(1-2a)} = 101$

2. Given $y = 3^x$.

    a) Write down the equation of the inverse of $y = 3^x$ in the form $y = \ldots$
    b) If $6 = 3^p$, determine the value of p (correct to one decimal place).
    c) Draw the graph of $y = 3^x$ and its inverse. Plot the points $A(p; 6)$ and $B(6; p)$.

3. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 28HF 1b. 28HG 1c. 28HH 1d. 28HJ 1e. 28HK 1f. 28HM
1g. 28HN 2. 28HP

Summary

EMCFW

- The logarithm of a number (x) with a certain base (a) is equal to the exponent (y), the value to which that certain base must be raised to equal the number (x). If $x = a^y$, then $y = \log_a(x)$, where $a > 0$, $a \neq 1$ and $x > 0$.
- Logarithms and exponentials are inverses of each other.
  $f(x) = \log_a x$ and $f^{-1}(x) = a^x$
- Common logarithm: $\log a$ means $\log_{10} a$
- The "LOG" function on your calculator uses a base of 10.
- Natural logarithm: $\ln$ uses a base of e.
- Special values:
  - $a^0 = 1 \qquad \log_a 1 = 0$
  - $a^1 = a \qquad \log_a a = 1$
- Logarithmic laws:
  - $\log_a xy = \log_a x + \log_a y \qquad (x > 0 \text{ and } y > 0)$
  - $\log_a \frac{x}{y} = \log_a x - \log_a y \qquad (x > 0 \text{ and } y > 0)$
  - $\log_a x^b = b \log_a x \qquad (x > 0)$
  - $\log_a x = \frac{\log_b x}{\log_b a} \qquad (b > 0 \text{ and } b \neq 1)$
- Special reciprocal applications:
  - $\log_a x = \frac{1}{\log_x a}$
  - $\log_a \frac{1}{x} = -\log_a x$

Exercise 2 – 18: Logarithms (ENRICHMENT ONLY)

1. State whether the following are true or false. If false, change the statement so that it is true.

   a) $\log t + \log d = \log(t + d)$
   b) If $p^q = r$, then $q = \log_r p$
   c) $\log \frac{A}{B} = \log A - \log B$
   d) $\log A - B = \frac{\log A}{\log B}$
   e) $\log_{\frac{1}{2}} x = -\log_2 x$
   f) $\log_k m = \frac{\log_p k}{\log_p m}$
   g) $\log_n \sqrt{b} = \frac{1}{2} \log_n b$
   h) $\log_p q = \frac{1}{\log_q p}$
   i) $2\log_2 a + 3\log a = 5\log a$
   j) $5\log x + 10\log x = 5\log x^3$
   k) $\frac{\log_n a}{\log_n b} = \log_n \frac{a}{b}$
   l) $\log(A + B) = \log A + \log B$
   m) $\log 2a^3 = 3\log 2a$
   n) $\frac{\log_n a}{\log_n b} = \log_n(a - b)$

2. Simplify the following without using a calculator:

   a) $\log 7 - \log 0{,}7$
   b) $\log 8 \times \log 1$
   c) $\log \frac{1}{3} + \log 300$
   d) $2\log 3 + \log 2 - \log 6$

3. Given $\log 5 = 0{,}7$. Find the value of the following without using a calculator:

   a) $\log 50$
   b) $\log 20$
   c) $\log 25$
   d) $\log_2 5$
   e) $10^{0{,}7}$

4. Given $A = \log_8 1 - \log_5 \frac{1}{25} + \log_3 9$.

   a) Without using a calculator, show that $A = 4$.
   b) Now solve for x if $\log_2 x = A$.
   c) Let $f(x) = \log_2 x$. Draw the graph of f and f^{-1}. Indicate the point $(x; A)$ on the graph.

5. Solve for x if $\frac{35^x}{7^x} = 15$. Give answer correct to two decimal places.

6. Given $f(x) = 5 \times (1{,}5)^x$ and $g(x) = \left(\frac{1}{4}\right)^x$.

   a) For which integer values of x will $f(x) < 295$.
   b) For which values of x will $g(x) \geq 2{,}7 \times 10^{-7}$. Give answer to the nearest integer.

7. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 28HQ	1b. 28HR	1c. 28HS	1d. 28HT	1e. 28HV	1f. 28HW
1g. 28HX	1h. 28HY	1i. 28HZ	1j. 28J2	1k. 28J3	1l. 28J4
1m. 28J5	1n. 28J6	2a. 28J7	2b. 28J8	2c. 28J9	2d. 28JB
3a. 28JC	3b. 28JD	3c. 28JF	3d. 28JG	3e. 28JH	4. 28JJ
5. 28JK	6. 28JM				

CHAPTER

Finance

In earlier grades we studied simple interest and compound interest, together with the concept of depreciation. Nominal and effective interest rates were also described.

- Simple interest: $A = P(1 + in)$
- Compound interest: $A = P(1 + i)^n$
- Simple depreciation: $A = P(1 - in)$
- Compound depreciation: $A = P(1 - i)^n$
- Nominal and effective annual interest rates: $1 + i = \left(1 + \frac{i^{(m)}}{m}\right)^m$

In this chapter we look at different types of annuities, sinking funds and pyramid schemes. We also look at how to critically analyse investment and loan options and how to make informed financial decisions.

Financial planning is very important and it allows people to achieve certain goals, such as supporting a family, attending university, buying a house, and saving enough money for retirement. Prudent financial planning includes making a budget, opening a savings account, wisely investing savings and planning for retirement.

3.1 Calculating the period of an investment EMCFX

For calculations using the simple interest formula, we solve for n, the time period of an investment or loan, by simply rearranging the formula to make n the subject. For compound interest calculations, where n is an exponent in the formula, we need to use our knowledge of logarithms to determine the value of n.

$$A = P(1 + i)^n$$

$$\begin{aligned} A &= \text{accumulated amount} \\ P &= \text{principal amount} \\ i &= \text{interest rate written as a decimal} \\ n &= \text{time period} \end{aligned}$$

Solving for n:

$$\begin{aligned} A &= P(1 + i)^n \\ \frac{A}{P} &= (1 + i)^n \\ \text{Use definition: } n &= \log_{(1+i)}\left(\frac{A}{P}\right) \\ \text{Change of base: } n &= \frac{\log\left(\frac{A}{P}\right)}{\log(1 + i)} \end{aligned}$$

Worked example 1: Determining the value of n

QUESTION

Thembile invests R 3500 into a savings account which pays $7{,}5\%$ per annum compounded yearly. After an unknown period of time his account is worth R $4044{,}69$. For how long did Thembile invest his money?

SOLUTION

Step 1: Write down the compound interest formula and the known values

$$A = P(1+i)^n$$

$$\begin{aligned} A &= 4044{,}69 \\ P &= 3500 \\ i &= 0{,}075 \end{aligned}$$

Step 2: Substitute the values and solve for n

$$\begin{aligned} A &= P(1+i)^n \\ 4044{,}69 &= 3500(1+0{,}075)^n \\ \frac{4044{,}69}{3500} &= 1{,}075^n \\ \therefore n &= \log_{(1{,}075)}\left(\frac{4044{,}69}{3500}\right) \\ &= \frac{\log \frac{4044{,}69}{3500}}{\log 1{,}075} \\ &= 2{,}00\ldots \end{aligned}$$

Step 3: Write final answer

The R 3500 was invested for 2 years.

Worked example 2: Duration of investments

QUESTION

Margo has R $12\ 000$ to invest and needs the money to grow to at least R $30\ 000$ to pay for her daughter's studies. If it is invested at a compound interest rate of 9% per annum, determine how long (in full years) her money must be invested?

SOLUTION

Step 1: Write down the compound interest formula and the known values

$$A = P(1+i)^n$$

$$\begin{aligned} A &= 30\ 000 \\ P &= 12\ 000 \\ i &= 0{,}09 \end{aligned}$$

Step 2: Substitute the values and solve for n

$$\begin{aligned} A &= P(1+i)^n \\ 30\ 000 &= 12\ 000(1+0{,}09)^n \\ \frac{5}{2} &= (1{,}09)^n \\ \therefore n &= \log_{1{,}09}\left(\frac{5}{2}\right) \qquad \text{(use definition)} \\ &= \frac{\log\frac{5}{2}}{\log 1{,}09} \qquad \text{(change of base)} \\ &= 10{,}632\ldots \end{aligned}$$

Step 3: Write the final answer

In this case we round up, because 10 years will not yet deliver the required R $30\ 000$. Therefore the money must be invested for at least 11 years.

Exercise 3 – 1: Determining the period of an investment

1. Nzuzo invests R $80\ 000$ at an interest rate of $7{,}5\%$ per annum compounded yearly. How long will it take for his investment to grow to R $100\ 000$?
2. Sally invests R $120\ 000$ at an interest rate of 12% per annum compounded quarterly. How long will it take for her investment to double?
3. When Banele was still in high school he deposited R 2250 into a savings account with an interest rate of $6{,}99\%$ per annum compounded yearly. How long ago did Banele open the account if the balance is now R $2882{,}53$? Write the answer as a combination of years and months.
4. The annual rate of depreciation of a vehicle is 15%. A new vehicle costs R $122\ 000$. After how many years will the vehicle be worth less than R $40\ 000$?
5. Some time ago, a man opened a savings account at KMT South Bank and deposited an amount of R 2100. The balance of his account is now R $3160{,}59$. If the account gets $8{,}52\%$ compound interest p.a., determine how many years ago the man made the deposit.
6. Mr. and Mrs. Dlamini want to save money for their son's university fees. They deposit R 7000 in a savings account with a fixed interest rate of $6{,}5\%$ per year compounded annually. How long will take for this deposit to double in value?
7. A university lecturer retires at the age of 60. She has saved R $300\ 000$ over the years.
   a) She decides not to let her savings decrease at a rate faster than 15% per year. How old will she be when the value of her savings is less than R $50\ 000$?
   b) If she doesn't use her savings and invests all her money in an investment account that earns a fixed interest rate of $5{,}95\%$ per annum, how long will it take for her investment to grow to R $390\ 000$?

8. Simosethu puts R 450 into a bank account at the Bank of Upington. Simosethu's account pays interest at a rate of $7{,}11\%$ p.a. compounded monthly. After how many years will the bank account have a balance of R $619{,}09$?
9. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1\. 28JP 2. 28JQ 3. 28JR 4. 28JS 5. 28JT 6. 28JV
7\. 28JW 8. 28JX

3.2 Annuities EMCFY

DEFINITION: *Annuity*

A number of equal payments made at regular intervals for a certain amount of time. An annuity is subject to a rate of interest.

- **Future value annuity** - regular equal deposits/payments are made into a savings account or investment fund to provide an accumulated amount at the end of the time period. The amount accumulating in the fund earns compound interest at a certain rate.
- **Present value annuity** - regular equal payments/installments are made to pay back a loan or bond over a given time period. The reducing balance of the loan is usually charged compound interest at a certain rate.

For investment funds, pension funds, loan repayments, mortgage bonds (home loan) and other types of annuities, payments are typically made each month. To "default" on a payment means that a payment for a certain month was not paid. The period of an investment is also referred to as the term of an investment.

3.3 Future value annuities EMCFZ

For future value annuities, we regularly save the same amount of money into an account, which earns a certain rate of compound interest, so that we have money for the future.

Worked example 3: Future value annuities

QUESTION

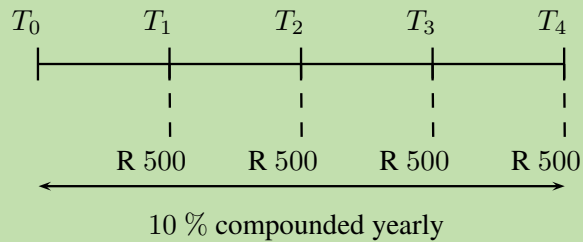
At the end of each year for 4 years, Kobus deposits R 500 into an investment account. If the interest rate on the account is 10% per annum compounded yearly, determine the value of his investment at the end of the 4 years.

SOLUTION

Step 1: Write down the given information and the compound interest formula

$$A = P(1+i)^n$$

$$P = 500$$
$$i = 0{,}1$$
$$n = 4$$

Step 2: Draw a timeline

The first deposit in the account earns the highest amount of interest (three interest payments) and the last deposit earns the least interest (no interest payments).

We can summarize this information in the table below:

	Deposit	No. of interest payments	Calculation	Accumulated amount
Year 1	R 500	3	$500(1+0{,}1)^3$	R 665,50
Year 2	R 500	2	$500(1+0{,}1)^2$	R 605,00
Year 3	R 500	1	$500(1+0{,}1)^1$	R 550,00
Year 4	R 500	0	$500(1+0{,}1)^0$	R 500,00
Total				R 2320,50

Deriving the formula EMCG2

Note: for this section is it important to be familiar with the formulae for the sum of a geometric series (Chapter 1):

$$S_n = \frac{a\left(r^n - 1\right)}{r-1} \qquad \text{for } r > 1$$

$$S_n = \frac{a\left(1 - r^n\right)}{1-r} \qquad \text{for } r < 1$$

In the worked example above, the total value of Kobus' investment at the end of the four year period is calculated by summing the accumulated amount for each deposit:

$$\begin{aligned}\text{R } 2320{,}50 &= \text{R } 500{,}00 \quad + \quad \text{R } 550{,}00 \quad + \quad \text{R } 605{,}00 \quad + \quad \text{R } 665{,}50 \\ &= 500(1+0{,}1)^0 + 500(1+0{,}1)^1 + 500(1+0{,}1)^2 + 500(1+0{,}1)^3\end{aligned}$$

We notice that this is a geometric series with a constant ratio $r = 1 + 0{,}1$.

Using the formula for the sum of a geometric series:

$$\begin{aligned}a &= 500 \\ r &= 1{,}1 \\ n &= 4\end{aligned}$$

$$\begin{aligned}S_n &= \frac{a\left(r^n - 1\right)}{r - 1} \\ &= \frac{500\left(1{,}1^4 - 1\right)}{1{,}1 - 1} \\ &= 2320{,}50\end{aligned}$$

We can therefore use the formula for the sum of a geometric series to derive a formula for the future value (F) of a series of (n) regular payments of an amount (x) which are subject to an interest rate (i):

$$\begin{aligned}a &= x \\ r &= 1 + i\end{aligned}$$

$$\begin{aligned}S_n &= \frac{a\left(r^n - 1\right)}{r - 1} \\ \therefore F &= \frac{x\left[(1+i)^n - 1\right]}{(1+i) - 1} \\ &= \frac{x\left[(1+i)^n - 1\right]}{i}\end{aligned}$$

Future value of payments:

$$F = \frac{x\left[(1+i)^n - 1\right]}{i}$$

If we are given the future value of a series of payments, then we can calculate the value of the payments by making x the subject of the above formula.

Payment amount:

$$x = \frac{F \times i}{\left[(1+i)^n - 1\right]}$$

Worked example 4: Future value annuities

QUESTION

Ciza decides to start saving money for the future. At the end of each month she deposits R 900 into an account at Harringstone Mutual Bank, which earns $8{,}25\%$ interest p.a. compounded monthly.

1. Determine the balance of Ciza's account after 29 years.
2. How much money did Ciza deposit into her account over the 29 year period?
3. Calculate how much interest she earned over the 29 year period.

SOLUTION

Step 1: Write down the given information and the future value formula

$$F = \frac{x\left[(1+i)^n - 1\right]}{i}$$

$$\begin{aligned} x &= 900 \\ i &= \frac{0{,}0825}{12} \\ n &= 29 \times 12 = 348 \end{aligned}$$

Step 2: Substitute the known values and use a calculator to determine F

$$\begin{aligned} F &= \frac{900\left[(1+\frac{0{,}0825}{12})^{348} - 1\right]}{\frac{0{,}0825}{12}} \\ &= \text{R } 1\ 289\ 665{,}06 \end{aligned}$$

Remember: do not round off at any of the interim steps of a calculation as this will affect the accuracy of the final answer.

Step 3: Calculate the total value of deposits into the account

Ciza deposited R 900 each month for 29 years:

$$\begin{aligned} \text{Total deposits: } &= \text{R } 900 \times 12 \times 29 \\ &= \text{R } 313\ 200 \end{aligned}$$

Step 4: Calculate the total interest earned

$$\begin{aligned} \text{Total interest } &= \text{final account balance } - \text{total value of all deposits} \\ &= \text{R } 1\ 289\ 665{,}06 - \text{R } 313\ 200 \\ &= \text{R } 976\ 465{,}06 \end{aligned}$$

Useful tips for solving problems:

1. Timelines are very useful for summarising the given information in a visual way.
2. When payments are made more than once per annum, we determine the total number of payments (n) by multiplying the number of years by p:

Term	p
yearly / annually	1
half-yearly / bi-annually	2
quarterly	4
monthly	12
weekly	52
daily	365

3. If a nominal interest rate $\left(i^{(m)}\right)$ is given, then use the following formula to convert it to an effective interest rate:
$$1+i=\left(1+\frac{i^{(m)}}{m}\right)^m$$

Worked example 5: Calculating the monthly payments

QUESTION

Kosma is planning a trip to Canada to visit her friend in two years' time. She makes an itinerary for her holiday and she expects that the trip will cost R $25\ 000$. How much must she save at the end of every month if her savings account earns an interest rate of $10{,}7\%$ per annum compounded monthly?

SOLUTION

Step 1: Write down the given information and the future value formula

$$F=\frac{x\left[(1+i)^n-1\right]}{i}$$

To determine the monthly payment amount, we make x the subject of the formula:

$$x=\frac{F\times i}{\left[(1+i)^n-1\right]}$$

$$F=25\ 000$$
$$i=\frac{0{,}107}{12}$$
$$n=2\times 12=24$$

Step 2: Substitute the known values and calculate x

$$x=\frac{25\ 000\times \frac{0{,}107}{12}}{\left[(1+\frac{0{,}107}{12})^{24}-1\right]}$$
$$=\text{R } 938{,}80$$

Step 3: Write the final answer

Kosma must save R $938{,}80$ each month so that she can afford her holiday.

Worked example 6: Determining the value of an investment

QUESTION

Simon starts to save for his retirement. He opens an investment account and immediately deposits R 800 into the account, which earns $12{,}5\%$ per annum compounded monthly. Thereafter, he deposits R 800 at the end of each month for 20 years. What is the value of his retirement savings at the end of the 20 year period?

SOLUTION

Step 1: Write down the given information and the future value formula

$$F = \frac{x\left[(1+i)^n - 1\right]}{i}$$

$$\begin{aligned} x &= 800 \\ i &= \frac{0{,}125}{12} \\ n &= 1 + (20 \times 12) = 241 \end{aligned}$$

Note that we added one extra month to the 20 years because Simon deposited R 800 immediately.

Step 2: Substitute the known values and calculate F

$$\begin{aligned} F &= \frac{800\left[(1+\frac{0{,}125}{12})^{241} - 1\right]}{\frac{0{,}125}{12}} \\ &= \text{R } 856\ 415{,}66 \end{aligned}$$

Step 3: Write the final answer

Simon will have saved R $856\ 415{,}66$ for his retirement.

Exercise 3 – 2: Future value annuities

1. Shelly decides to start saving money for her son's future. At the end of each month she deposits R 500 into an account at Durban Trust Bank, which earns an interest rate of $5{,}96\%$ per annum compounded quarterly.
   a) Determine the balance of Shelly's account after 35 years.
   b) How much money did Shelly deposit into her account over the 35 year period?
   c) Calculate how much interest she earned over the 35 year period.
2. Gerald wants to buy a new guitar worth R 7400 in a year's time. How much must he deposit at the end of each month into his savings account, which earns a interest rate of $9{,}5\%$ p.a. compounded monthly?

3. A young woman named Grace has just started a new job, and wants to save money for the future. She decides to deposit R 1100 into a savings account every month. Her money goes into an account at First Mutual Bank, and the account earns $8{,}9\%$ interest p.a. compounded every month.

    a) How much money will Grace have in her account after 29 years?

    b) How much money did Grace deposit into her account by the end of the 29 year period?

4. Ruth decides to save for her retirement so she opens a savings account and immediately deposits R 450 into the account. Her savings account earns 12% per annum compounded monthly. She then deposits R 450 at the end of each month for 35 years. What is the value of her retirement savings at the end of the 35 year period?

5. Musina MoneyLenders offer a savings account with an interest rate of $6{,}13\%$ p.a. compounded monthly. Monique wants to save money so that she can buy a house when she retires. She decides to open an account and make regular monthly deposits. Her goal is to end up with R $750\ 000$ in her account after 35 years.

    a) How much must Monique deposit into her account each month in order to reach her goal?

    b) How much money, to the nearest rand, did Monique deposit into her account by the end of the 35 year period?

6. Lerato plans to buy a car in five and a half years' time. She has saved R $30\ 000$ in a separate investment account which earns 13% per annum compound interest. If she doesn't want to spend more than R $160\ 000$ on a vehicle and her savings account earns an interest rate of 11% p.a. compounded monthly, how much must she deposit into her savings account each month?

7. a) Every Monday Harold puts R 30 into a savings account at the King Bank, which accrues interest of $6{,}92\%$ p.a. compounded weekly. How long will it take Harold's account to reach a balance of R $4397{,}53$. Give the answer as a number of years and days to the nearest integer.

    b) How much interest will Harold receive from the bank during the period of his investment?

8. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 28JY   1b. 28JZ   1c. 28K2   2. 28K3   3. 28K4   4. 28K5
5. 28K6   6. 28K7   7. 28K8

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Sinking funds   EMCG3

Vehicles, equipment, machinery and other similar assets, all depreciate in value as a result of usage and age. Businesses often set aside money for replacing outdated equipment or old vehicles in accounts called sinking funds. Regular deposits, and sometimes lump sum deposits, are made into these accounts so that enough money will have accumulated by the time a new machine or vehicle needs to be purchased.

Worked example 7: Sinking funds

QUESTION

Wellington Courier Company buys a delivery truck for R 296 000. The value of the truck depreciates on a reducing-balance basis at 18% per annum. The company plans to replace this truck in seven years' time and they expect the price of a new truck to increase annually by 9%.

1. Calculate the book value of the delivery truck in seven years' time.
2. Determine the minimum balance of the sinking fund in order for the company to afford a new truck in seven years' time.
3. Calculate the required monthly deposits if the sinking fund earns an interest rate of 13% per annum compounded monthly.

SOLUTION

Step 1: Determine the book value of the truck in seven years' time

$$\begin{aligned} P &= 296\ 000 \\ i &= 0{,}18 \\ n &= 7 \end{aligned}$$

$$\begin{aligned} A &= P(1-i)^n \\ &= 296\ 000(1-0{,}18)^7 \\ &= \text{R } 73\ 788{,}50 \end{aligned}$$

Step 2: Determine the minimum balance of the sinking fund

Calculate the price of a new truck in seven years' time:

$$\begin{aligned} P &= 296\ 000 \\ i &= 0{,}09 \\ n &= 7 \end{aligned}$$

$$\begin{aligned} A &= P(1+i)^n \\ &= 296\ 000(1+0{,}09)^7 \\ &= \text{R } 541\ 099{,}58 \end{aligned}$$

Therefore, the balance of the sinking fund (F) must be greater than the cost of a new truck in seven years' time minus the money from the sale of the old truck:

$$\begin{aligned} F &= \text{R } 541\ 099{,}58 - \text{R } 73\ 788{,}50 \\ &= \text{R } 467\ 311{,}08 \end{aligned}$$

Step 3: Calculate the required monthly payment into the sinking fund

$$x = \frac{F \times i}{[(1+i)^n - 1]}$$

$$\begin{aligned} F &= 467\ 311{,}08 \\ i &= \frac{0{,}13}{12} \\ n &= 7 \times 12 = 84 \end{aligned}$$

Substitute the values and calculate x:

$$x = \frac{467\,311{,}08 \times \frac{0{,}13}{12}}{\left[(1 + \frac{0{,}13}{12})^{84} - 1\right]}$$
$$= \text{R } 3438{,}77$$

Therefore, the company must deposit R $3438{,}77$ each month.

Exercise 3 – 3: Sinking funds

1. Mfethu owns his own delivery business and he will need to replace his truck in 6 years' time. Mfethu deposits R 3100 into a sinking fund each month, which earns $5{,}3\%$ interest p.a. compounded monthly.
   a) How much money will be in the fund in 6 years' time, when Mfethu wants to buy the new truck?
   b) If a new truck costs R $285\ 000$ in 6 years' time, will Mfethu have enough money to buy it?
2. Atlantic Transport Company buys a van for R $265\ 000$. The value of the van depreciates on a reducing-balance basis at 17% per annum. The company plans to replace this van in five years' time and they expect the price of a new van to increase annually by 12%.
   a) Calculate the book value of the van in five years' time.
   b) Determine the amount of money needed in the sinking fund for the company to be able to afford a new van in five years' time.
   c) Calculate the required monthly deposits if the sinking fund earns an interest rate of 11% per annum compounded monthly.
3. Tonya owns Freeman Travel Company and she will need to replace her computer in 7 years' time. Tonya creates a sinking fund so that she will be able to afford a new computer, which will cost R 8450. The sinking fund earns interest at a rate of $7{,}67\%$ p.a. compounded each quarter.
   a) How much money must Tonya save quarterly so that there will be enough money in the account to buy the new computer?
   b) How much interest (to the nearest rand) does the bank pay into the account by the end of the 7 year period?
4. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 28K9 2. 28KB 3. 28KC

3.4 Present value annuities EMCG4

For present value annuities, regular equal payments/installments are made to pay back a loan or bond over a given time period. The reducing balance of the loan is usually charged compound interest at a certain rate. In this section we learn how to determine the present value of a series of payments.

Consider the following example:

Kate needs to withdraw R 1000 from her bank account every year for the next three years. How much must she deposit into her account, which earns 10% per annum, to be able to make these withdrawals in the future? We will assume that these are the only withdrawals and that there are no bank charges on her account.

To calculate Kate's deposit, we make P the subject of the compound interest formula:

$$\begin{aligned} A &= P(1+i)^n \\ \frac{A}{(1+i)^n} &= P \\ \therefore P &= A(1+i)^{-n} \end{aligned}$$

We determine how much Kate must deposit for the first withdrawal:

$$\begin{aligned} P &= 1000(1+0{,}1)^{-1} \\ &= 909{,}09 \end{aligned}$$

We repeat this calculation to determine how much must be deposited for the second and third withdrawals:

$$\begin{aligned} \text{Second withdrawal: } P &= 1000(1+0{,}1)^{-2} \\ &= 826{,}45 \\ \text{Third withdrawal: } P &= 1000(1+0{,}1)^{-3} \\ &= 751{,}31 \end{aligned}$$

Notice that for each year's withdrawal, the deposit required gets smaller and smaller because it will be in the bank account for longer and therefore earn more interest. Therefore, the total amount is:

$$\text{R } 909{,}09 + \text{R } 826{,}45 + \text{R } 751{,}31 = \text{R } 2486{,}85$$

We can check these calculations by determining the accumulated amount in Kate's bank account after each withdrawal:

	Calculation	Accumulated amount
Initial deposit		R $2486{,}85$
Amount after one year	$= 2486{,}85(1+0{,}1)$	$=$ R $2735{,}54$
Amount after first withdrawal	$=$ R $2735{,}54 -$ R 1000	$=$ R $1735{,}54$
Amount after two years	$= 1735{,}54(1+0{,}1)$	$=$ R $1909{,}09$
Amount after second withdrawal	$=$ R $1909{,}09 -$ R 1000	$=$ R $909{,}09$
Amount after three years	$= 909{,}09(1+0{,}1)$	$=$ R 1000
Amount after third withdrawal	$=$ R $1000 -$ R 1000	$=$ R 0

Completing this table for a three year period does not take too long. However, if Kate needed to make annual payments for 20 years, then the calculation becomes very repetitive and time-consuming. Therefore, we need a more efficient method for performing these calculations.

Deriving the formula EMCG5

In the example above, Kate needed to deposit:

$$\begin{aligned} \text{R } 2486{,}55 &= \text{R } 909{,}09 \quad + \quad \text{R } 826{,}45 \quad + \quad \text{R } 751{,}31 \\ &= 1000(1+0{,}1)^{-1} + 1000(1+0{,}1)^{-2} + 1000(1+0{,}1)^{-3} \end{aligned}$$

We notice that this is a geometric series with a constant ratio $r = (1+0{,}1)^{-1}$.

Using the formula for the sum of a geometric series:

$$\begin{aligned} a &= 1000(1+0{,}1)^{-1} \\ r &= (1+0{,}1)^{-1} \\ n &= 3 \end{aligned}$$

$$\begin{aligned} S_n &= \frac{a\left(1-r^n\right)}{1-r} \qquad (\text{for } r<1) \\ &= \frac{1000(1+0{,}1)^{-1}\left[1-\left((1+0{,}1)^{-1}\right)^3\right]}{1-(1+0{,}1)^{-1}} \\ &= \frac{1000\left[1-(1+0{,}1)^{-3}\right]}{(1+0{,}1)[1-(1+0{,}1)^{-1}]} \\ &= \frac{1000\left[1-(1+0{,}1)^{-3}\right]}{(1+0{,}1)-1} \\ &= \frac{1000\left[1-(1+0{,}1)^{-3}\right]}{0{,}1} \\ &= 2486{,}85 \end{aligned}$$

We can therefore use the formula for the sum of a geometric series to derive a formula for the present value (P) of a series of (n) regular payments of an amount (x) which are subject to an interest rate (i):

$$a = x(1+i)^{-1}$$
$$r = (1+i)^{-1}$$

$$S_n = \frac{a(1-r^n)}{1-r} \qquad \text{(for } r < 1\text{)}$$

$$\therefore P = \frac{x(1+i)^{-1}\left[1-((1+i)^{-1})^n\right]}{1-(1+i)^{-1}}$$

$$= \frac{x\left[1-(1+i)^{-n}\right]}{(1+i)[1-(1+i)^{-1}]}$$

$$= \frac{x\left[1-(1+i)^{-n}\right]}{1+i-1}$$

$$= \frac{x\left[1-(1+i)^{-n}\right]}{i}$$

Present value of a series of payments:

$$P = \frac{x\left[1-(1+i)^{-n}\right]}{i}$$

If we are given the present value of a series of payments, we can calculate the value of the payments by making x the subject of the above formula.

Payment amount:

$$x = \frac{P \times i}{[1-(1+i)^{-n}]}$$

Worked example 8: Present value annuities

QUESTION

Andre takes out a student loan for his first year of civil engineering. The loan agreement states that the repayment period is equal to $1{,}5$ years for every year of financial assistance granted and that the loan is subject to an interest rate of $10{,}5\%$ p.a. compounded monthly.

1. If Andre pays a monthly installment of R $1446{,}91$, calculate the loan amount.
2. Determine how much interest Andre will have paid on his student loan at the end of the 18 months.

SOLUTION

Step 1: Write down the given information and the present value formula

$$P = \frac{x\left[1-(1+i)^{-n}\right]}{i}$$

$$\begin{aligned} x &= 1446{,}91 \\ i &= \frac{0{,}105}{12} \\ n &= 1{,}5 \times 12 = 18 \end{aligned}$$

Substitute the known values and determine P:

$$\begin{aligned} P &= \frac{1446{,}91\left[1-(1+\frac{0{,}105}{12})^{-18}\right]}{\frac{0{,}105}{12}} \\ &= \text{R } 24\ 000{,}14 \end{aligned}$$

Therefore, Andre took out a student loan for R $24\ 000$.

Step 2: Calculate the total amount of interest

At the end of the 18 month period:

$$\begin{aligned} \text{Total amount repaid for loan: } &= \text{R } 1446{,}91 \times 18 \\ &= \text{R } 26\ 044{,}38 \\ \text{Total amount of interest: } &= \text{R } 26\ 044{,}38 - \text{R } 24\ 000 \\ &= \text{R } 2044{,}38 \end{aligned}$$

Worked example 9: Calculating the monthly payments

QUESTION

Hristo wants to buy a small wine farm worth R $8\ 500\ 000$. He plans to sell his current home for R $3\ 400\ 000$ which he will use as a deposit for the purchase of the farm. He secures a loan with HBP Bank with a repayment period of 10 years and an interest rate of $9{,}5\%$ compounded monthly.

1. Calculate his monthly repayments.
2. Determine how much interest Hristo will have paid on his loan by the end of the 10 years.

SOLUTION

Step 1: Write down the given information and the present value formula

$$P = \frac{x\left[1-(1+i)^{-n}\right]}{i}$$

To determine the monthly repayment amount, we make x the subject of the formula:

$$x = \frac{P \times i}{[1 - (1 + i)^{-n}]}$$

$$\begin{aligned} P &= \text{R } 8\ 500\ 000 - \text{R } 3\ 400\ 000 = \text{R } 5\ 100\ 000 \\ i &= \frac{0{,}095}{12} \\ n &= 10 \times 12 = 120 \end{aligned}$$

Step 2: Substitute the known values and calculate x

$$\begin{aligned} x &= \frac{5\ 100\ 000 \times \frac{0{,}095}{12}}{\left[1 - \left(1 + \frac{0{,}095}{12}\right)^{-120}\right]} \\ &= \text{R } 65\ 992{,}75 \end{aligned}$$

Therefore, Hristo must pay R $65\ 992{,}75$ per month to repay his loan over the 10 year period.

Step 3: Calculate the total amount of interest
At the end of the 10 year period:

$$\begin{aligned} \text{Total amount repaid for loan: } &= \text{R } 65\ 992{,}75 \times 10 \times 12 \\ &= \text{R } 7\ 919\ 130 \\ \text{Total amount of interest: } &= \text{R } 7\ 919\ 130 - \text{R } 5\ 100\ 000 \\ &= \text{R } 2\ 819\ 130 \end{aligned}$$

Prime lending rate

The prime lending rate is a benchmark rate at which private banks lend out money to the public. It is used as a reference rate for determining interest rates on many types of loans, including small business loans, home loans and personal loans. Some interest rates may be expressed as a percentage above or below prime rate. For calculations in this chapter, we will assume the prime lending rate is $8{,}5\%$ per annum.

Worked example 10: Calculating the outstanding balance of a loan

QUESTION

A school sells its old bus and uses the proceeds as a 15% deposit for the purchase of the new bus, which costs R $330\ 000$. To finance the balance of the purchase, the school takes out a loan that is subject to an interest rate of prime $+1\%$ compounded monthly. The repayment period of the loan is 3 years.

1. Calculate the monthly repayments.
2. Determine the balance of the loan at the end of the first year, immediately after the 12^{th} payment.

SOLUTION

Step 1: Write down the given information and the present value formula

$$P = \frac{x\left[1 - (1+i)^{-n}\right]}{i}$$

To determine the monthly repayment amount, we make x the subject of the formula:

$$x = \frac{P \times i}{\left[1 - (1+i)^{-n}\right]}$$

$$\begin{aligned}
P &= \text{R } 330\ 000 - \left(\frac{15}{100} \times \text{R } 330\ 000\right) \\
&= \text{R } 330\ 000 - \text{R } 49\ 500 \\
&= \text{R } 280\ 500 \\
i &= \frac{0{,}095}{12} \\
n &= 3 \times 12 = 36
\end{aligned}$$

Step 2: Substitute the known values and calculate x

$$\begin{aligned}
x &= \frac{280\ 500 \times \frac{0{,}095}{12}}{\left[1 - \left(1 + \frac{0{,}095}{12}\right)^{-36}\right]} \\
&= \text{R } 8985{,}24
\end{aligned}$$

Therefore, the school must pay R $8985{,}24$ per month to repay the loan over the 3 year period.

Step 3: Calculate the balance of the loan at the end of the first year

We can calculate the balance of the loan at the end of the first year by determining the present value of the remaining 24 payments:

$$\begin{aligned}
P &= \frac{8985{,}24\left[1 - (1 + \frac{0{,}095}{12})^{-24}\right]}{\frac{0{,}095}{12}} \\
&= \text{R } 195\ 695{,}07
\end{aligned}$$

Therefore, the school must still pay R $195\ 695{,}07$ of the loan.

Alternative method: we can also calculate the balance of the loan at the end of the first year by determining the accumulated amount of the loan for the first year less the future value of the first 12 payments:

$$\begin{aligned}
\text{Balance} &= A(\text{loan and accrued interest}) - F(\text{first year's payments and interest}) \\
&= 280\ 500\left(1 + \frac{0{,}095}{12}\right)^{12} - \frac{8985{,}24\left[(1 + \frac{0{,}095}{12})^{12} - 1\right]}{\frac{0{,}095}{12}} \\
&= \text{R } 308\ 338{,}94\ldots - \text{R } 112\ 643{,}79\ldots \\
&= \text{R } 195\ 695{,}15
\end{aligned}$$

Therefore, the school must still pay R $195\ 695{,}15$ of the loan.

(Note the difference of 8 cents due to rounding).

Exercise 3 – 4: Present value annuities

1. A property costs R $1\ 800\ 000$. Calculate the monthly repayments if the interest rate is 14% p.a. compounded monthly and the loan must be paid off in 20 years' time.
2. A loan of R 4200 is to be returned in two equal annual installments. If the rate of interest is 10% compounded annually, calculate the amount of each installment.
3. Stefan and Marna want to buy a flat that costs R $1{,}2$ million. Their parents offer to put down a 20% payment towards the cost of the house. They need to get a mortgage for the balance. What is the monthly repayment amount if the term of the home loan is 30 years and the interest is $7{,}5\%$ p.a. compounded monthly?
4. a) Ziyanda arranges a bond for R $17\ 000$ from Langa Bank. If the bank charges $16{,}0\%$ p.a. compounded monthly, determine Ziyanda's monthly repayment if she is to pay back the bond over 9 years.
   b) What is the total cost of the bond?
5. Dullstroom Bank offers personal loans at an interest rate of $15{,}63\%$ p.a. compounded twice a year. Lubabale borrows R 3000 and must pay R $334{,}93$ every six months until the loan is fully repaid.
   a) How long will it take Lubabale to repay the loan?
   b) How much interest will Lubabale pay for this loan?
6. Likengkeng has just started a new job and wants to buy a car that costs R $232\ 000$. She visits the Soweto Savings Bank, where she can arrange a loan with an interest rate of $15{,}7\%$ p.a. compounded monthly. Likengkeng has enough money saved to pay a deposit of R $50\ 000$. She arranges a loan for the balance of the payment, which is to be paid over a period of 6 years.
   a) What is Likengkeng's monthly repayment on her loan?
   b) How much will the car cost Likengkeng?
7. Anathi is a wheat farmer and she needs to buy a new holding tank which costs R $219\ 450$. She bought her old tank 14 years ago for R $196\ 000$. The value of the old grain tank has depreciated at a rate of $12{,}1\%$ per year on a reducing balance, and she plans to trade it in for its current value. Anathi will then need to arrange a loan for the balance of the cost of the new grain tank.

   Orsmond bank offers loans with an interest rate of $9{,}71\%$ p.a. compounded monthly for any loan up to R $170\ 000$ and $9{,}31\%$ p.a. compounded monthly for a loan above that amount. The loan agreement allows Anathi a grace period for the first six months (no payments are made) and it states that the loan must be repaid over 30 years.
   a) Determine the monthly repayment amount.
   b) What is the total amount of interest Anathi will pay for the loan?
   c) How much money would Anathi have saved if she did not take the six month grace period?
8. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 28KD 2. 28KF 3. 28KG 4. 28KH 5. 28KJ 6. 28KK
7. 28KM

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3.5 Analysing investment and loan options EMCG6

In the next worked example we consider the effects of the duration of the repayment period on the total amount repaid (the amount borrowed plus the accrued interest) for a loan.

Worked example 11: Repayment periods

QUESTION

David and Julie take out a home loan of R $2{,}6$ million with an interest rate of 10% per annum compounded monthly.

1. Calculate the monthly repayments for a repayment period of 30 years.
2. Calculate the interest paid on the loan at the end of the 30 year period.
3. Determine the monthly repayments for a repayment period of 20 years.
4. Determine the interest paid on the loan at the end of the 20 year period.
5. What is the difference in the monthly repayment amounts?
6. Comment on the difference in the interest paid for the two different time periods.

SOLUTION

Step 1: Consider a 30 year repayment period on the loan

$$x = \frac{P \times i}{[1 - (1 + i)^{-n}]}$$

$$\begin{aligned} P &= \text{R } 2\ 600\ 000 \\ i &= \frac{0{,}1}{12} \\ n &= 30 \times 12 = 360 \end{aligned}$$

$$\begin{aligned} x &= \frac{2\ 600\ 000 \times \frac{0{,}1}{12}}{\left[1 - \left(1 + \frac{0{,}1}{12}\right)^{-360}\right]} \\ &= \text{R } 22\ 816{,}86 \end{aligned}$$

Therefore, the monthly repayment is R $22\ 816{,}86$ for a 30 year period.

At the end of the 30 years, David and Julie will have paid a total amount of:

$$\begin{aligned} &= 30 \times 12 \times \text{R } 22\ 816{,}86 \\ &= \text{R } 8\ 214\ 069{,}60 \end{aligned}$$

The total amount of interest on the loan:

$$\begin{aligned} \text{Interest} &= \text{total amount paid} - \text{loan amount} \\ &= \text{R } 8\ 214\ 069{,}60 - \text{R } 2\ 600\ 000 \\ &= \text{R } 5\ 614\ 069{,}60 \end{aligned}$$

We notice that the interest on the loan is more than double the amount borrowed.

Step 2: Consider a 20 year repayment period on the loan

$$x = \frac{P \times i}{[1-(1+i)^{-n}]}$$

$$\begin{aligned} P &= \text{R } 2\ 600\ 000 \\ i &= \frac{0,1}{12} \\ n &= 20 \times 12 = 240 \end{aligned}$$

$$\begin{aligned} x &= \frac{2\ 600\ 000 \times \frac{0,1}{12}}{\left[1-\left(1+\frac{0,1}{12}\right)^{-240}\right]} \\ &= \text{R } 25\ 090,56 \end{aligned}$$

Therefore, the monthly repayment is R $25\ 090{,}56$ for a 20 year period.

At the end of the 20 years, David and Julie will have paid a total amount of:

$$\begin{aligned} &= 20 \times 12 \times \text{R } 25\ 090,56 \\ &= \text{R } 6\ 021\ 734,40 \end{aligned}$$

The total amount of interest on the loan:

$$\begin{aligned} \text{Interest} &= \text{total amount paid} - \text{loan amount} \\ &= \text{R } 6\ 021\ 734,40 - \text{R } 2\ 600\ 000 \\ &= \text{R } 3\ 421\ 734,40 \end{aligned}$$

We notice that the interest on the loan is about $1{,}3$ times the borrowed amount.

Step 3: Consider the difference in the repayment and interest amounts

$$\begin{aligned} \text{Difference in repayments } &= \text{R } 25\ 090,56 - \text{R } 22\ 816,86 \\ &= \text{R } 2273,70 \end{aligned}$$

It is also very interesting to look at the difference in the total interest paid:

$$\begin{aligned} \text{Difference in interest } &= \text{R } 5\ 614\ 069,60 - \text{R } 3\ 421\ 734,40 \\ &= \text{R } 2\ 192\ 335,20 \end{aligned}$$

Therefore, by paying an extra R $2273{,}70$ each month over a shorter repayment period, David and Julie could save more than R 2 million on the repayment of their home loan.

When considering taking out a loan, it is advisable to investigate and compare a few options offered by financial institutions. It is very important to make informed decisions regarding personal finances and to make sure that the monthly repayment amount

is serviceable (payable). A credit rating is an estimate of a person's ability to fulfill financial commitments based on their previous payment history. Defaulting on a loan can affect a person's credit rating and their chances of taking out another loan in the future.

Worked example 12: Analysing investment opportunities

QUESTION

Marlene wants to start saving for a deposit on a house. She can afford to invest between R 400 and R 600 each month and gets information from four different investment firms. Each firm quotes a different interest rate and a prescribed monthly installment amount. She plans to buy a house in 7 years' time. Calculate which company offers the best investment opportunity for Marlene.

	Interest rate (compounded monthly)	Monthly payment
TBS Investments	$13{,}5\%$ p.a.	R 450
Taylor Anderson	13% p.a.	R 555
PHK	$12{,}5\%$ p.a.	R 575
Simfords Consulting	11% p.a.	R 600

SOLUTION

Step 1: Consider the different investment options

To compare the different investment options, we need to calculate the following for each option at the end of the seven year period:

- The future value of the monthly payments.
- The total amount paid into the investment fund.
- The total interest earned.

$$F = \frac{x\left[(1+i)^n - 1\right]}{i}$$

TBS Investments:

$$\begin{aligned} F &= \frac{450\left[(1+\frac{0{,}135}{12})^{84} - 1\right]}{\frac{0{,}135}{12}} \\ &= \text{R } 62\ 370{,}99 \\ \text{Total amount } (T): &= 7 \times 12 \times \text{R } 450 \\ &= \text{R } 37\ 800 \\ \text{Total interest } (I): &= \text{R } 62\ 370{,}99 - \text{R } 37\ 800 \\ &= \text{R } 24\ 570{,}99 \end{aligned}$$

Taylor Anderson:

$$
\begin{aligned}
F &= \frac{555\left[(1+\frac{0,13}{12})^{84}-1\right]}{\frac{0,13}{12}} \\
&= \text{R } 75\ 421,65 \\
\text{Total amount } (T): &= 7 \times 12 \times \text{R } 555 \\
&= \text{R } 46\ 620 \\
\text{Total interest } (I): &= \text{R } 75\ 421,65 - \text{R } 46\ 620 \\
&= \text{R } 28\ 801,65
\end{aligned}
$$

PHK:

$$
\begin{aligned}
F &= \frac{575\left[(1+\frac{0,125}{12})^{84}-1\right]}{\frac{0,125}{12}} \\
&= \text{R } 76\ 619,96 \\
\text{Total amount } (T): &= 7 \times 12 \times \text{R } 575 \\
&= \text{R } 48\ 300 \\
\text{Total interest } (I): &= \text{R } 76\ 619,96 - \text{R } 48\ 300 \\
&= \text{R } 28\ 319,96
\end{aligned}
$$

Simfords Consulting:

$$
\begin{aligned}
F &= \frac{600\left[(1+\frac{0,11}{12})^{84}-1\right]}{\frac{0,11}{12}} \\
&= \text{R } 75\ 416,96 \\
\text{Total amount } (T): &= 7 \times 12 \times \text{R } 600 \\
&= \text{R } 50\ 400 \\
\text{Total interest } (I): &= \text{R } 75\ 416,96 - \text{R } 50\ 400 \\
&= \text{R } 25\ 016,96
\end{aligned}
$$

Step 2: Draw a table of the results to compare the answers

	F	T	I
TBS Investments	R 62 370,99	R 37 800	R 24 570,99
Taylor Anderson	R 75 421,65	R 46 620	R 28 801,65
PHK	R 76 619,96	R 48 300	R 28 319,96
Simfords Consulting	R 75 416,96	R 50 400	R 25 016,96

Step 3: Make a conclusion

An investment with PHK would provide Marlene with the highest deposit (R 76 619,96) for her house at the end of the 7 year period. However, we notice that an investment with Taylor Anderson would earn the highest amount of interest (R 28 801,65) and is therefore the better investment option.

Worked example 13: Analysing loan options

QUESTION

William wants to take out a loan of R $750\ 000$, so he approaches three different banks. He plans to start repaying the loan immediately and he calculates that he can afford a monthly repayment amount between R 5500 and R 7000.

Calculate which of the three options would be best for William.

- West Bank offers a repayment period of 30 years and an interest rate of prime compounded monthly.
- AcuBank offers a repayment period of 20 years and an interest rate of prime $+0{,}5\%$ compounded monthly.
- FinTrust Bank offers a repayment period of 15 years and an interest rate of prime $+2\%$ compounded monthly.

SOLUTION

Step 1: Consider the different loan options

To compare the different loan options, we need to calculate the following for each option:

- The monthly payment amount.
- The total amount paid to repay the loan.
- The amount of interest on the loan.

$$x = \frac{P \times i}{[1 - (1+i)^{-n}]}$$

West Bank:

$$\begin{aligned} x &= \frac{750\ 000 \times \frac{0{,}085}{12}}{\left[1 - (1 + \frac{0{,}085}{12})^{-360}\right]} \\ &= \text{R } 5766{,}85 \\ \text{Total amount } (T): &= 30 \times 12 \times \text{R } 5766{,}85 \\ &= \text{R } 2\ 076\ 066 \\ \text{Total interest } (I): &= \text{R } 2\ 076\ 066 - \text{R } 750\ 000 \\ &= \text{R } 1\ 326\ 066 \end{aligned}$$

AcuBank:

$$\begin{aligned} x &= \frac{750\ 000 \times \frac{0{,}09}{12}}{\left[1 - (1 + \frac{0{,}09}{12})^{-240}\right]} \\ &= \text{R } 6747{,}94 \\ \text{Total amount } (T): &= 20 \times 12 \times \text{R } 6747{,}94 \\ &= \text{R } 1\ 619\ 505{,}60 \\ \text{Total interest } (I): &= \text{R } 1\ 619\ 505{,}60 - \text{R } 750\ 000 \\ &= \text{R } 869\ 505{,}60 \end{aligned}$$

FinTrust Bank:

$$x = \frac{750\ 000 \times \frac{0,105}{12}}{\left[1-(1+\frac{0,105}{12})^{-180}\right]}$$
$$= \text{R } 8290,49$$

$$\text{Total amount } (T): = 15 \times 12 \times \text{R } 8290,49$$
$$= \text{R } 1\ 492\ 288,20$$
$$\text{Total interest } (I): = \text{R } 1\ 492\ 288,20 - \text{R } 750\ 000$$
$$= \text{R } 742\ 288,20$$

Step 2: Draw a table of the results to compare the answers

	x	T	I
West Bank	R 5766,85	R 2 076 066,00	R 1 326 066,00
AcuBank	R 6747,94	R 1 619 505,60	R 869 505,60
FinTrust Bank	R 8290,49	R 1 492 288,20	R 742 288,20

Step 3: Make a conclusion

A loan from FinTrust Bank would accumulate the lowest amount of interest but the monthly repayment amounts are not within William's budget. Although West Bank offers the lowest interest rate and monthly repayment amount, the interest earned on the loan is very high as a result of the longer repayment period. If we assume that William must repay the loan over the given time periods, then AcuBank offers the best option.

However, we know that William can afford to pay more than R 5766,85 per month, and if the bank allows him to pay back the loan earlier, he should consider taking out a loan with West Bank and take advantage of the lower interest rate.

Exercise 3 – 5: Analysing investment and loan options

1. Cokisa is 31 years old and starting to plan for her future. She has been thinking about her retirement and wants to open an annuity so that she will have money when she retires. Her intention is to retire when she is 65 years old. Cokisa visits the Trader's Bank of Tembisa and learns that there are two investment options from which she can choose:
   - Option A: 7,76% p.a. compounded once every four months
   - Option B: 7,78% p.a. compounded half-yearly

   a) Which is the better investment option for Cokisa if the amount she will deposit will always be the same?

   b) Cokisa opens an account and starts saving R 4000 every four months. How much money (to the nearest rand) will she have saved when she reaches her planned retirement?

2. Phoebe wants to take out a home loan of R $1,6$ million. She approaches three different banks for their loan options:

   - Bank A offers a repayment period of 30 years and an interest rate of 12% per annum compounded monthly.
   - Bank B offers a repayment period of 20 years and an interest rate of 14% per annum compounded monthly.
   - Bank C offers a repayment period of 30 years and an interest rate of 14% per annum compounded monthly.

   If Phoebe intends to start her monthly repayments immediately, calculate which of the three options would be best for her.
3. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 28KN 2. 28KP

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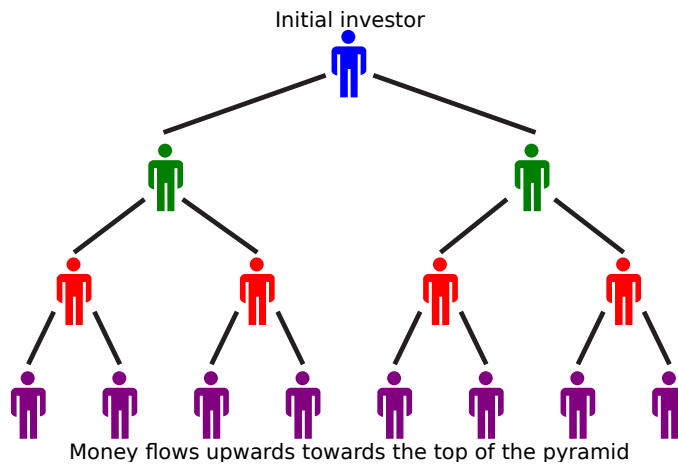
Pyramid schemes EMCG7

A pyramid scheme is a moneymaking scheme that promises investors unusually high returns on their investment. The concept of a pyramid scheme is quite simple and should be easy to identify, however, it is often cleverly disguised as a legitimate business. In some cases a product is offered and in others the scheme is marketed as a highly profitable investment opportunity. Unfortunately, many of these schemes have cost millions of people their savings. Pyramid schemes are illegal in South Africa.

- "Ponzi schemes" are named after Charles Ponzi, a Italian businessman who lived in the U.S. and Canada. One of his investment schemes involved buying postal reply coupons in other countries and redeeming them for a higher value in the United States. He promised investors 100% profit within 90 days of their investment. As the scheme grew, Ponzi paid off early investors with money from investors who joined the scheme at a later stage. Ponzi's fraudulent scheme was exposed and investors lost millions of dollars. He was sent to prison for a number of years.
- South African Adriaan Nieuwoudt started a pyramid scheme commonly referred to as the "Kubus" scheme. Participants bought a biological substance called an activator that was supposed to be used in beauty products. The activator was used to grow cultures in milk that were then dried and ground up and resold to new participants. The scheme had thousands of investors and took in approximately R 140 million before it was declared an illegal lottery.

A pyramid scheme starts with one person, who finds other people to invest their money in the scheme. These people then enrol more people into the scheme and the base of the pyramid grows. The money invested by new investors goes to participants closer to the top of the pyramid. This is unsustainable because it requires that more and more people join the scheme and the growth must end at some point because there are a finite number of people. Therefore, most of the investors lose their money when the scheme collapses.

The South African Reserve Bank has launched a public awareness campaign, "Beware of oMashayana (crooks)", to help educate people about pyramid schemes and how to identify them.

> ***Beware of oMashayana***
>
> *So, you think you've found the perfect investment? Regular high returns with no risk? Be careful. Don't lose all your money. If it sounds too good to be true, it's probably a Pyramid or Ponzi scheme.*
>
> ***What is a Ponzi scheme?***
>
> *A Ponzi con-artist will only ask you to give them money that they say will be invested in a scheme or project, such as supposed property developments; bridging finance; foreign-exchange transactions; venture capital to other companies; or share units. The scheme operator promises to give you back much more money than you have given them initially, in a very short space of time.*
>
> ***What is a pyramid scheme?***
>
> *A pyramid con-artist will offer you the chance to make quick money for yourself, often by selling something. You pay a joining fee, buy the product and then sell it. They tell you that the more people you get to sell for you, the more money you will make. It's easy for the first people to introduce new members, but soon everyone is part of the scheme and it gets harder to find new members to join. For example, the con-artist recruits* 6 *people who each pay a R* 100 *fee. Each of those people has to recruit* 6 *people. There are now up to* 36 *people. Now, those* 36 *people each have to sign up* 6 *people – this equals* 216*, and by the time you have to get to level* 10*, you have to get* 60 *million people to join to keep the scheme going. This can never work.*
>
> ***What is the difference between a pyramid and a Ponzi scheme?***
>
> *The main difference is that with a pyramid scheme you have to work or sell to recruit investors, while with a Ponzi scheme the con-artist will only ask you to invest in something (for example, property development). Both schemes are illegal.*

What can I do to protect myself?

You are your own best protection. It is your responsibility to make sure that you never give your money to any company or person that is not registered as a deposit-taking institution in terms of the Banks Act.

Why is handing my money over so risky?

When you hand over your money (notes and coins) to another person who then loses the money, steals it or goes bankrupt, you only have an unsecured claim against that person or their estate and you might not get all your money back.

Why is it safer to hand my money to a bank?

Banks and investment companies have to be registered so that they can be regulated and supervised, to make sure that your money is safe. Unregulated and unsupervised persons and groups don't follow these rules and your money is at great risk with them.

3.6 Summary EMCG8

- Always keep the rate of interest per time unit and the time period in the same units.
- Simple interest: $A = P(1 + in)$
- Compound interest: $A = P(1 + i)^n$
- Simple depreciation: $A = P(1 - in)$
- Compound depreciation: $A = P(1 - i)^n$
- Nominal and effective annual interest rates: $1 + i = \left(1 + \frac{i^{(m)}}{m}\right)^m$
- **Future value of payments:**

$$F = \frac{x\left[(1 + i)^n - 1\right]}{i}$$

Payment amount:

$$x = \frac{F \times i}{\left[(1 + i)^n - 1\right]}$$

- **Present value of a series of payments:**

$$P = \frac{x\left[1 - (1 + i)^{-n}\right]}{i}$$

Payment amount:

$$x = \frac{P \times i}{\left[1 - (1 + i)^{-n}\right]}$$

Exercise 3 – 6: End of chapter exercises

1. Mpumelelo deposits R 500 into a savings account, which earns interest at $6{,}81\%$ p.a. compounded quarterly. How long will it take for the savings account to have a balance of R $749{,}77$?
2. How much interest will Gavin pay on a loan of R $360\ 000$ for 5 years at $10{,}3\%$ per annum compounded monthly?
3. Wingfield school will need to replace a number of old classroom desks in 6 years' time. The principal has calculated that the new desks will cost R $44\ 500$. The school establishes a sinking fund to pay for the new desks and immediately deposits an amount of R 6300 into the fund, which accrues interest at a rate of $6{,}85\%$ p.a. compounded monthly.
   a) How much money should the school save every month so that the sinking fund will have enough money to cover the cost of the desks?
   b) How much interest does the fund earn over the period of 6 years?
4. Determine how many years (to the nearest integer) it will take for the value of a motor vehicle to decrease to 25% of its original value if the rate of depreciation, based on the reducing-balance method, is 21% per annum.
5. Angela has just started a new job, and wants to save money for her retirement. She decides to deposit R 1300 into a savings account once each month. Her money goes into an account at Pinelands Mutual Bank, and the account receives $6{,}01\%$ interest p.a. compounded once each month.
   a) How much money will Angela have in her account after 30 years?
   b) How much money did Angela deposit into her account after 30 years?
6. a) Nicky has been working at Meyer and Associates for 5 years and gets an increase in her salary. She opens a savings account at Langebaan Bank and begins making deposits of R 350 every month. The account earns $5{,}53\%$ interest p.a., compounded monthly. Her plan is to continue saving on a monthly schedule until she retires. However, after 8 years she stops making the monthly payments and leaves the account to continue growing.

      How much money will Nicky have in her account 29 years after she first opened it?
   b) Calculate the difference between the total deposits made into the account and the amount of interest paid by the bank.
7. a) Every three months Louis puts R 500 into an annuity. His account earns an interest rate of $7{,}51\%$ p.a. compounded quarterly. How long will it take Louis's account to reach a balance of R $13\ 465{,}87$?
   b) How much interest will Louis receive from his investment?
8. A dairy farmer named Kayla needs to buy new equipment for her dairy farm which costs R $200\ 450$. She bought her old equipment 12 years ago for R $167\ 000$. The value of the old equipment depreciates at a rate of $12{,}2\%$ per year on a reducing balance. Kayla will need to arrange a bond for the remaining cost of the new equipment.

   An agency which supports farmers offers bonds at a special interest rate of $10{,}01\%$ p.a. compounded monthly for any loan up to R $175\ 000$ and $9{,}61\%$ p.a. compounded monthly for a loan above that amount. Kayla arranges a bond such that she will not need to make any payments on the loan in the first six months (called a 'grace period') and she must pay the loan back over 20 years.
   a) Determine the monthly payment.
   b) What is the total amount of interest Kayla will pay for the bond?

c) By what factor is the interest she pays greater than the value of the loan? Give the answer correct to one decimal place.

9. Thabo invests R 8500 in a special banking product which will pay 1% per annum for 1 month, then 2% per annum for the next 2 months, then 3% per annum for the next 3 months, 4% per annum for the next 4 months, and 0% for the rest of the year. If the bank charges him R 75 to open the account, how much can he expect to get back at the end of the year?
10. Thabani and Lungelo are both using Harper Bank for their savings. Lungelo makes a deposit of x at an interest rate of i for six years. Three years after Lungelo made his first deposit, Thabani makes a deposit of $3x$ at an interest rate of 8% per annum. If after 6 years their investments are equal, calculate the value of i (correct to three decimal places). If the sum of their investment is R $20\ 000$, determine how much Thabani earned in 6 years.
11. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

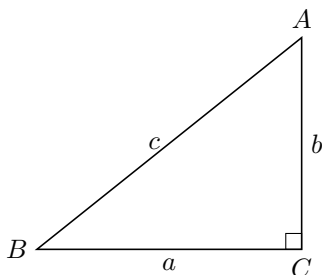
1. 28KQ 2. 28KR 3. 28KS 4. 28KT 5. 28KV 6. 28KW
7. 28KX 8. 28KY 9. 28KZ 10. 28M2

CHAPTER 4

Trigonometry

4.1 Revision EMCG9

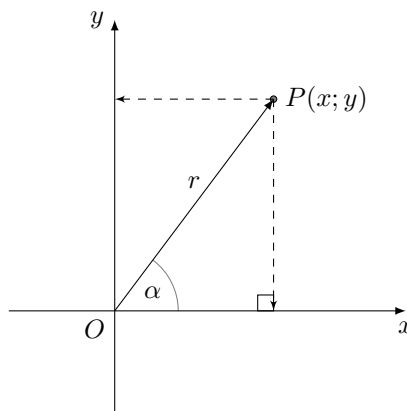
Trigonometric ratios

We defined the basic trigonometric ratios using the lengths of the sides of a right-angled triangle.

$$\sin \hat{A} = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{a}{c} \qquad \sin \hat{B} = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{b}{c}$$

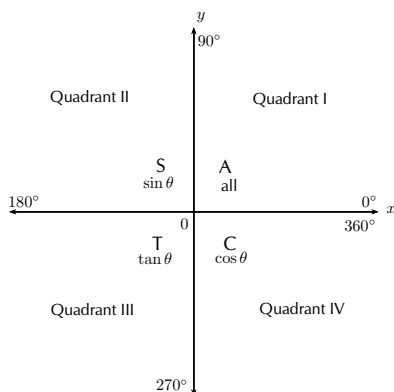
$$\cos \hat{A} = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{b}{c} \qquad \cos \hat{B} = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{a}{c}$$

$$\tan \hat{A} = \frac{\text{opposite}}{\text{adjacent}} = \frac{a}{b} \qquad \tan \hat{B} = \frac{\text{opposite}}{\text{adjacent}} = \frac{b}{a}$$

Trigonometric ratios in the Cartesian plane

We also defined the trigonometric ratios with respect to any point in the Cartesian plane in terms of x, y and r. Using the theorem of Pythagoras, $r^2 = x^2 + y^2$.

$$\sin \alpha = \frac{y}{r} \qquad \cos \alpha = \frac{x}{r} \qquad \tan \alpha = \frac{y}{x}$$

CAST diagram

The sign of a trigonometric ratio depends on the signs of x and y:

Reduction formulae and co-functions:

1. The reduction formulae hold for any angle θ. For convenience, we assume θ is an acute angle ($0° < \theta < 90°$).
2. When determining function values of ($180° \pm \theta$), ($360° \pm \theta$) and ($-\theta$) the function does not change.
3. When determining function values of ($90° \pm \theta$) and ($\theta \pm 90°$) the function changes to its co-function.

y

Second Quadrant:

- **sine function is positive**
- $\sin(180° - \theta) = \sin\theta$
- $\cos(180° - \theta) = -\cos\theta$
- $\tan(180° - \theta) = -\tan\theta$
- $\sin(90° + \theta) = \cos\theta$
- $\cos(90° + \theta) = -\sin\theta$

First Quadrant:

- **all functions are positive**
- $\sin(360° + \theta) = \sin\theta$
- $\cos(360° + \theta) = \cos\theta$
- $\tan(360° + \theta) = \tan\theta$
- $\sin(90° - \theta) = \cos\theta$
- $\cos(90° - \theta) = \sin\theta$

O x

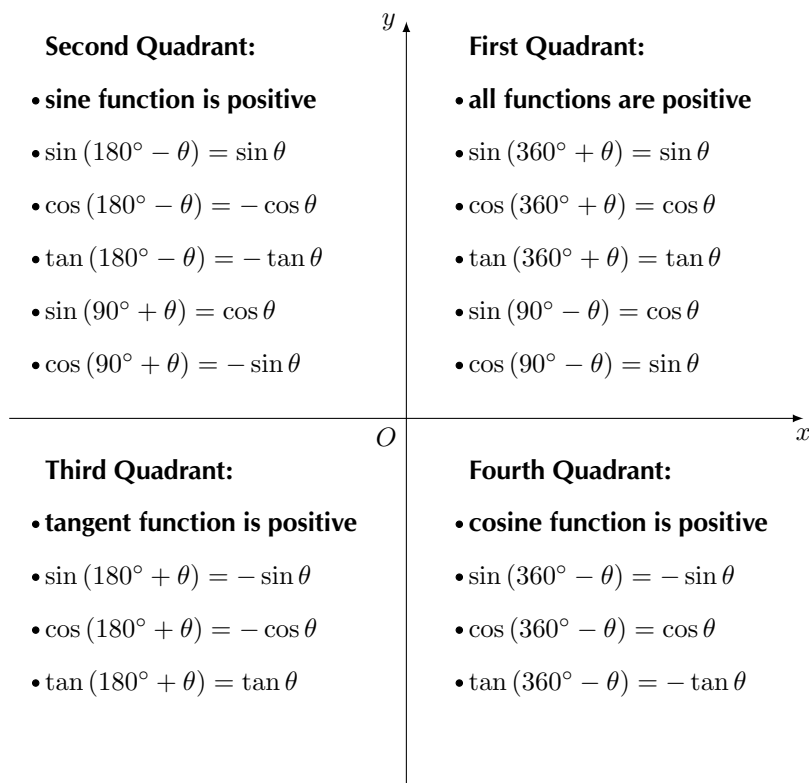
Third Quadrant:

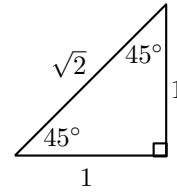
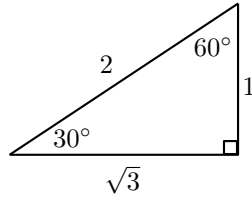
- **tangent function is positive**
- $\sin(180° + \theta) = -\sin\theta$
- $\cos(180° + \theta) = -\cos\theta$
- $\tan(180° + \theta) = \tan\theta$

Fourth Quadrant:

- **cosine function is positive**
- $\sin(360° - \theta) = -\sin\theta$
- $\cos(360° - \theta) = \cos\theta$
- $\tan(360° - \theta) = -\tan\theta$

Negative angles

$$\begin{aligned} \sin(-\theta) &= -\sin\theta \\ \cos(-\theta) &= \cos\theta \\ \tan(-\theta) &= -\tan\theta \end{aligned}$$

Special angle triangles

These values are useful when we need to solve a problem involving trigonometric functions without using a calculator. Remember that the lengths of the sides of a right-angled triangle obey the theorem of Pythagoras.

θ	$0°$	$30°$	$45°$	$60°$	$90°$
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undef

Trigonometric identities

Quotient identity:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad (\cos\theta \neq 0)$$

Square identity:

$$\sin^2\theta + \cos^2\theta = 1$$

It also follows that:

$$\begin{aligned} \sin^2\theta &= 1 - \cos^2\theta \\ \cos^2\theta &= 1 - \sin^2\theta \end{aligned}$$

$$\begin{aligned} \sin\theta &= \pm\sqrt{1 - \cos^2\theta} \\ \cos\theta &= \pm\sqrt{1 - \sin^2\theta} \end{aligned}$$

All these relationships and identities are very useful for simplifying trigonometric expressions.

Worked example 1: Revision

QUESTION

Determine the value of the expression, without using a calculator:

$$\frac{\cos 420° - \sin 225° \cos(-45°)}{\tan 315°}$$

SOLUTION

Step 1: Use reduction formulae to express each trigonometric ratio in terms of an acute angle

$$\begin{aligned}
&\frac{\cos 420° - \sin 225° \cos(-45°)}{\tan 315°} \\
&= \frac{\cos(360° + 60°) - \sin(180° + 45°) \cos(-45°)}{\tan(360° - 45°)} \\
&= \frac{\cos 60° - (-\sin 45°)(\cos 45°)}{-\tan 45°} \\
&= \frac{\cos 60° + \sin 45° \cos 45°}{-\tan 45°}
\end{aligned}$$

Now use special angles to evaluate the simplified expression:

$$\begin{aligned}
&= \frac{\cos 60° + \sin 45° \cos 45°}{-\tan 45°} \\
&= \frac{\frac{1}{2} + \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)}{-1} \\
&= -\left(\frac{1}{2} + \frac{1}{2}\right) \\
&= -1
\end{aligned}$$

Worked example 2: Revision

QUESTION

Prove:

$$\sin^2 \alpha - (\tan \alpha - \cos \alpha)(\tan \alpha + \cos \alpha) = \frac{\cos^2 \alpha - \sin^2 \alpha}{\cos^2 \alpha}$$

State restrictions where applicable.

SOLUTION

Step 1: Use trigonometric identities to simplify each side separately

Simplify the left-hand side of the identity:

$$\begin{aligned}
\text{LHS} &= \sin^2 \alpha - (\tan \alpha - \cos \alpha)(\tan \alpha + \cos \alpha) \\
&= \sin^2 \alpha - (\tan^2 \alpha - \cos^2 \alpha) \\
&= \sin^2 \alpha - \tan^2 \alpha + \cos^2 \alpha \\
&= \left(\sin^2 \alpha + \cos^2 \alpha\right) - \tan^2 \alpha \\
&= 1 - \tan^2 \alpha
\end{aligned}$$

Simplify the right-hand side of the identity so that it equals the left-hand side:

$$\begin{aligned}\text{RHS} &= \frac{\cos^2 \alpha - \sin^2 \alpha}{\cos^2 \alpha} \\ &= \frac{\cos^2 \alpha}{\cos^2 \alpha} - \frac{\sin^2 \alpha}{\cos^2 \alpha} \\ &= 1 - \tan^2 \alpha\end{aligned}$$

$$\therefore \text{LHS} = \text{RHS}$$

Alternative method: we could also have started with the left-hand side of the identity and substituted $\tan \alpha = \frac{\sin \alpha}{\cos \alpha}$ and simplified to get the right-hand side.

Restrictions

We need to determine the values of α for which any of the terms in the identity will be undefined:

$$\begin{aligned}\cos^2 \alpha &= 0 \\ \therefore \cos \alpha &= 0 \\ \therefore \alpha &= 90° \text{ or } 270°\end{aligned}$$

We must also consider the values of α for which $\tan \alpha$ is undefined. Therefore, the identity is undefined for $\alpha = 90° + k \,.\, 180°$.

Useful tips:

- It is sometimes useful to write $\tan \theta$ in terms of $\sin \theta$ and $\cos \theta$.
- Never write a trigonometric ratio without an angle. For example, $\tan = \frac{\sin}{\cos}$ has no meaning.
- For proving identities, only simplify one side of the identity at a time.
- As seen in the worked example above, sometimes both sides of the identity need to be simplified.
- Remember to write down restrictions:
  - the values for which any of the trigonometric ratios are not defined;
  - the values of the variable which make any of the denominators in the identity equal to zero.

Exercise 4 – 1: Revision - reduction formulae, co-functions and identities

1. Given: $\sin 31° = A$

   Write each of the following expressions in terms of A:

   a) $\sin 149°$

   b) $\cos(-59°)$

   c) $\cos 329°$

   d) $\tan 211° \cos 211°$

   e) $\tan 31°$

2. a) Simplify P to a single trigonometric ratio:

$$P = \sin(360° + \theta)\cos(180° + \theta)\tan(360° + \theta)$$

b) Simplify Q to a single trigonometric ratio:

$$Q = \frac{\cos(\theta - 360°)\sin(90° + \theta)\sin(-\theta)}{\sin(\theta + 180°)}$$

c) Hence, determine:

i. $P + Q$

ii. $\frac{Q}{P}$

3. If $p = \sin\beta$, express the following in terms of p:

$$\frac{\cos(\beta + 360°)\tan(\beta - 360°)\cos(\beta + 90°)}{\sin^2(\beta + 180°)\cos(\beta - 90°)}$$

4. Evaluate the following without the use of a calculator:

a) $\frac{\cos(-120°)}{\tan 150°} + \cos 390°$

b) $(1 - \sin 45°)(1 - \sin 225°)$

5. Reduce the following to one trigonometric ratio:

a) $\tan^2\beta - \frac{1}{\cos^2\beta}$

b) $\sin^2(90° + \theta)\tan^2\theta + \tan^2\theta\cos^2(90° - \theta)$

c) $\sin\alpha\cos\alpha\tan\alpha - 1$

d) $\tan^2\theta + \frac{\cos^2\theta - 1}{\cos^2\theta}$

6. a) Use reduction formulae and special angles to show that

$$\frac{\sin(180° + \theta)\tan(720° + \theta)\cos(-\theta)}{\cos(90° + \theta)}$$

can be simplified to $\sin\theta$.

b) Without using a calculator, determine the value of $\sin 570°$.

7. Troy's mathematics teacher asks the class to answer the following question.

Question:

Prove that $\frac{\cos\theta}{1 + \sin\theta} = \frac{1 - \sin\theta}{\cos\theta}$.

Troy's answer:

$$\frac{\cos\theta}{1 + \sin\theta} = \frac{1 - \sin\theta}{\cos\theta}$$
$$(\cos\theta)(\cos\theta) = (1 + \sin\theta)(1 - \sin\theta)$$
$$\cos^2\theta = 1 - \sin^2\theta$$
$$\cos^2\theta = \cos^2\theta$$
$$\therefore \text{LHS} = \text{RHS}$$

Comment on Troy's answer and show the correct method for proving this identity.

8. Prove the following identities:
(State any restricted values in the interval $[0°; 360°]$, where appropriate.)

a) $\sin^2\alpha + (\cos\alpha - \tan\alpha)(\cos\alpha + \tan\alpha) = 1 - \tan^2\alpha$

b) $\frac{1}{\cos\theta} - \frac{\cos\theta\tan^2\theta}{1} = \cos\theta$

c) $\frac{2\sin\theta\cos\theta}{\sin\theta + \cos\theta} = \sin\theta + \cos\theta - \frac{1}{\sin\theta + \cos\theta}$

d) $\left(\frac{\cos\beta}{\sin\beta} + \tan\beta\right)\cos\beta = \frac{1}{\sin\beta}$

e) $\frac{1}{1 + \sin\theta} + \frac{1}{1 - \sin\theta} = \frac{2\tan\theta}{\sin\theta\cos\theta}$

f) $(1 + \tan^2\alpha)\cos\alpha = \frac{1 - \tan\alpha}{\cos\alpha - \sin\alpha}$

9. Determine whether the following statements are true or false.
If the statement is false, choose a suitable value between $0°$ and $90°$ to confirm your answer.

a) $\cos(180° - \theta) = -1 - \cos\theta$

b) $\sin(\alpha + \beta) = \sin\alpha + \sin\beta$

c) $\sin\alpha = 2\sin\frac{\alpha}{2}\sin\frac{\alpha}{2}$

d) $\frac{1}{3}\sin 3\alpha = \sin\alpha$

e) $\cos\beta = \sqrt{1 - \sin^2\beta}$

f) $\sin\theta = \tan\theta\cos\theta$

10. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 28M3	1b. 28M4	1c. 28M5	1d. 28M6	1e. 28M7	2. 28M8
3. 28M9	4a. 28MB	4b. 28MC	5a. 28MD	5b. 28MF	5c. 28MG
5d. 28MH	6. 28MJ	7. 28MK	8a. 28MM	8b. 28MN	8c. 28MP
8d. 28MQ	8e. 28MR	8f. 28MS	9a. 28MT	9b. 28MV	9c. 28MW
9d. 28MX	9e. 28MY	9f. 28MZ			

4.2 Compound angle identities EMCGB

Derivation of $\cos(\alpha - \beta)$ EMCGC

Investigation: Compound angles

Danny is studying for a trigonometry test and completes the following question:

Question:

Evaluate the following: $\cos(180° - 120°)$

Danny's solution:

$$\begin{aligned} \cos(180° - 120°) &= \cos 180° - \cos 120° && \text{(line 1)} \\ &= -1 - \cos(90° + 30°) && \text{(line 2)} \\ &= -1 + \sin 30° && \text{(line 3)} \\ &= -1 + \tfrac{1}{2} && \text{(line 4)} \\ &= -\tfrac{1}{2} && \text{(line 5)} \end{aligned}$$

1. Consider Danny's solution and determine why it is incorrect.
2. Use a calculator to check that Danny's answer is wrong.
3. Describe in words the mistake(s) in his solution.
4. Is the following statement true or false?
   "A trigonometric ratio can be distributed to the angles that lie within the brackets."

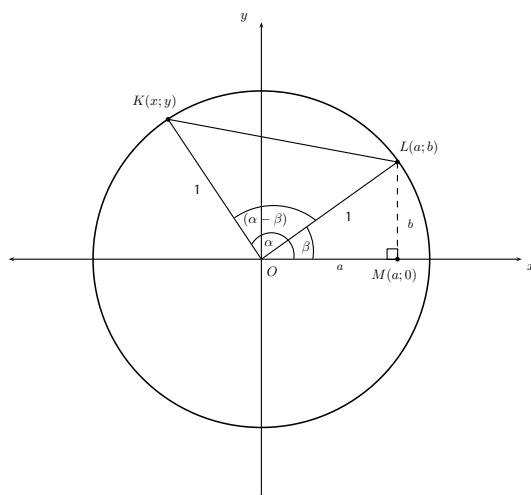
From the investigation above, we know that $\cos(\alpha - \beta) \neq \cos\alpha - \cos\beta$. It is wrong to apply the distributive law to the trigonometric ratios of compound angles.

Distance formula: $d_{AB} = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2}$

Cosine rule: $a^2 = b^2 + c^2 - 2bc \;.\; \cos\hat{A}$

Using the distance formula and the cosine rule, we can derive the following identity for compound angles:

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

Consider the unit circle $(r = 1)$ below. The two points $L\,(a; b)$ and $K\,(x; y)$ are shown on the circle.

We can express the coordinates of L and K in terms of the angles α and β:

$$\begin{aligned}\text{In } \triangle LOM, \quad \sin\beta &= \frac{b}{1}\\ \therefore b &= \sin\beta\\ \cos\beta &= \frac{a}{1}\\ \therefore a &= \cos\beta\\ L &= (\cos\beta; \sin\beta)\\ \text{Similarly, } K &= (\cos\alpha; \sin\alpha)\end{aligned}$$

We use the distance formula to determine KL^2:

$$\begin{aligned}d^2 &= (x_K - x_L)^2 + (y_K - y_L)^2\\ KL^2 &= (\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2\\ &= \cos^2\alpha - 2\cos\alpha\cos\beta + \cos^2\beta + \sin^2\alpha - 2\sin\alpha\sin\beta + \sin^2\beta\\ &= \left(\cos^2\alpha + \sin^2\alpha\right) + \left(\cos^2\beta + \sin^2\beta\right) - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta\\ &= 1 + 1 - 2\left(\cos\alpha\cos\beta + \sin\alpha\sin\beta\right)\\ &= 2 - 2\left(\cos\alpha\cos\beta + \sin\alpha\sin\beta\right)\end{aligned}$$

Now we determine KL^2 using the cosine rule for $\triangle KOL$:

$$\begin{aligned}KL^2 &= KO^2 + LO^2 - 2\,.\,KO\,.\,LO\,.\,\cos(\alpha - \beta)\\ &= 1^2 + 1^2 - 2(1)(1)\cos(\alpha - \beta)\\ &= 2 - 2\,.\,\cos(\alpha - \beta)\end{aligned}$$

Equating the two expressions for KL^2, we have

$$\begin{aligned}2 - 2\,.\,\cos(\alpha - \beta) &= 2 - 2\left(\cos\alpha\cos\beta + \sin\alpha\sin\beta\right)\\ 2\,.\,\cos(\alpha - \beta) &= 2\left(\cos\alpha\cos\beta + \sin\alpha\sin\beta\right)\\ \therefore \cos(\alpha - \beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta\end{aligned}$$

Worked example 3: Derivation of $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

QUESTION

Derive an expression for $\cos(\alpha + \beta)$ in terms of the trigonometric ratios of α and β.

SOLUTION

Step 1: Use the compound angle formula for $\cos(\alpha - \beta)$

We use the compound angle formula for $\cos(\alpha - \beta)$ and manipulate the sign of β in $\cos(\alpha + \beta)$ so that it can be written as a difference of two angles:

$$\begin{aligned}\cos(\alpha + \beta) &= \cos(\alpha - (-\beta))\\ \text{And we have shown } \cos(\alpha - \beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta\\ \therefore \cos[\alpha - (-\beta)] &= \cos\alpha\cos(-\beta) + \sin\alpha\sin(-\beta)\\ \therefore \cos(\alpha + \beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta\end{aligned}$$

Step 2: Write the final answer

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

Worked example 4: Derivation of $\sin(\alpha - \beta)$ and $\sin(\alpha + \beta)$

QUESTION

Derive the expanded formulae for $\sin(\alpha - \beta)$ and $\sin(\alpha + \beta)$ in terms of the trigonometric ratios of α and β.

SOLUTION

Step 1: Use the compound angle formula and co-functions to expand $\sin(\alpha - \beta)$

Using co-functions, we know that $\sin \hat{A} = \cos(90° - \hat{A})$, so we can write $\sin(\alpha + \beta)$ in terms of the cosine function as:

$$\begin{aligned} \sin(\alpha - \beta) &= \cos(90° - (\alpha - \beta)) \\ &= \cos(90° - \alpha + \beta) \\ &= \cos[(90° - \alpha) + \beta] \end{aligned}$$

Apply the compound angle formula:

$$\begin{aligned} \cos(\alpha + \beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \therefore \cos[(90° - \alpha) + \beta] &= \cos(90° - \alpha)\cos\beta - \sin(90° - \alpha)\sin\beta \\ \therefore \sin(\alpha - \beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta \end{aligned}$$

To derive the formula for $\sin(\alpha + \beta)$, we use the compound formula for $\sin(\alpha - \beta)$ and manipulate the sign of β:

$$\begin{aligned} \sin(\alpha - \beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta \\ \text{We can write } \sin(\alpha + \beta) &= \sin[\alpha - (-\beta)] \\ \therefore \sin[\alpha - (-\beta)] &= \sin\alpha\cos(-\beta) - \cos\alpha\sin(-\beta) \\ \therefore \sin(\alpha + \beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \end{aligned}$$

Step 2: Write the final answers

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

Compound angle formulae

- $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$
- $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
- $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$
- $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

Note: we can use the compound angle formulae to expand and simplify compound angles in trigonometric expressions (using the equations from left to right) or we can use the expanded form to determine the trigonometric ratio of a compound angle (using the equations from right to left).

Worked example 5: Compound angle formulae

QUESTION

Prove that $\sin 75^\circ = \frac{\sqrt{2}(\sqrt{3}+1)}{4}$ without using a calculator.

SOLUTION

Step 1: Consider the given identity

We know the values of the trigonometric functions for the special angles (30°, 45°, 60°, etc.) and we can write $75^\circ = 30^\circ + 45^\circ$.

Therefore, we can use the compound angle formula for $\sin(\alpha + \beta)$ to express $\sin 75^\circ$ in terms of known trigonometric function values.

Step 2: Prove the left-hand side of the identity equals the right-hand side

When proving an identity is true, remember to only work with one side of the identity at a time.

$$\begin{aligned}
\text{LHS} &= \sin 75^\circ \\
&= \sin(45^\circ + 30^\circ) \\
\sin(45^\circ + 30^\circ) &= \sin(45^\circ)\cos(30^\circ) + \cos(45^\circ)\sin(30^\circ) \\
&= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2} \\
&= \frac{\sqrt{3}+1}{2\sqrt{2}} \\
&= \frac{\sqrt{3}+1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \\
&= \frac{\sqrt{2}\left(\sqrt{3}+1\right)}{4} \\
&= \text{RHS}
\end{aligned}$$

Therefore, we have shown that $\sin 75^\circ = \frac{\sqrt{2}(\sqrt{3}+1)}{4}$.

Worked example 6: Compound angle formulae

QUESTION

Determine the value of the following expression without the use of a calculator:

$$\cos 65^\circ \cos 35^\circ + \cos 25^\circ \cos 55^\circ$$

SOLUTION

Step 1: Use co-functions to simplify the expression

- We need to change two of the trigonometric functions from cosine to sine so that we can apply the compound angle formula.

- We also need to make sure that the sum (or difference) of the two angles is equal to a special angle so that we can determine the value of the expression without using a calculator. Notice that $35° + 25° = 60°$.

$$\begin{aligned} &\cos 65° \cos 35° + \cos 25° \cos 55° \\ &= \cos(90° - 25°) \cos 35° + \cos 25° \cos(90° - 35°) \\ &= \sin 25° \cos 35° + \cos 25° \sin 35° \end{aligned}$$

Step 2: Apply the compound angle formula and use special angles to evaluate the expression

$$\begin{aligned} &\sin 25° \cos 35° + \cos 25° \sin 35° \\ &= \sin(25° + 35°) \\ &= \sin 60° \\ &= \frac{\sqrt{3}}{2} \end{aligned}$$

Step 3: Write the final answer

$$\cos 65° \cos 35° + \cos 25° \cos 55° = \frac{\sqrt{3}}{2}$$

Checking answers: It is always good to check answers. The question stated that we could not use a calculator to find the answer, but we can use a calculator to check that the answer is correct:

$$\begin{aligned} &\text{LHS} = \cos 65° \cos 35° + \cos 25° \cos 55° = 0{,}866\ldots \\ &\text{RHS} = \frac{\sqrt{3}}{2} = 0{,}866\ldots \\ \therefore\ &\text{LHS} = \text{RHS} \end{aligned}$$

Exercise 4 – 2: Compound angle formulae

1. Given:

$$13 \sin \alpha + 5 = 0 \qquad (0° < \alpha < 270°)$$

$$13 \cos \beta - 12 = 0 \qquad (90° < \beta < 360°)$$

   Draw a sketch and determine the following, without the use of a calculator:

   a) $\tan \alpha - \tan \beta$
   b) $\sin(\beta - \alpha)$
   c) $\cos(\alpha + \beta)$

2. Calculate the following without the use of a calculator (leave answers in surd form):

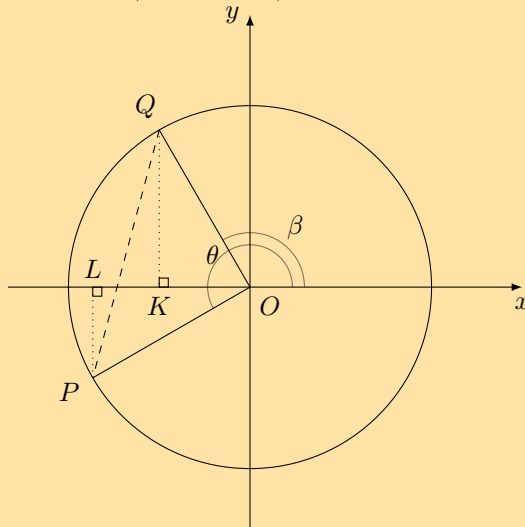
   a) $\sin 105°$
   b) $\cos 15°$
   c) $\sin 15°$
   d) $\tan 15°$
   e) $\cos 20° \cos 40° - \sin 20° \sin 40°$
   f) $\sin 10° \cos 80° + \cos 10° \sin 80°$
   g) $\cos(45° - x) \cos x - \sin(45° - x) \sin x$
   h) $\cos^2 15° - \sin^2 15°$

3. a) Prove: $\sin(60° - x) + \sin(60° + x) = \sqrt{3} \cos x$
   b) Hence, evaluate $\sin 15° + \sin 105°$ without using a calculator.
   c) Use a calculator to check your answer.

4. Simplify the following without using a calculator:

$$\frac{\sin p \cos(45° - p) + \cos p \sin(45° - p)}{\cos p \cos(60° - p) - \sin p \sin(60° - p)}$$

5. a) Prove: $\sin(A + B) - \sin(A - B) = 2 \cos A \sin B$
   b) Hence, calculate the value of $\cos 75° \sin 15°$ without using a calculator.

6. In the diagram below, points P and Q lie on the circle with radius of 2 units and centre at the origin.
   Prove $\cos(\theta - \beta) = \cos \theta \cos \beta + \sin \theta \sin \beta$.

7. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 28N3 2a. 28N4 2b. 28N5 2c. 28N6 2d. 28N7 2e. 28N8
2f. 28N9 2g. 28NB 2h. 28NC 3. 28ND 4. 28NF 5. 28NG
6. 28NH

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4.3 Double angle identities EMCGD

Derivation of $\sin 2\alpha$ EMCGF

We have shown that $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$. If we let $\alpha = \beta$, then we can write the formula as:

$$\begin{aligned} \sin(2\alpha) &= \sin(\alpha+\alpha) \\ &= \sin\alpha\cos\alpha + \cos\alpha\sin\alpha \\ \therefore \sin 2\alpha &= 2\sin\alpha\cos\alpha \end{aligned}$$

Derivation of $\cos 2\alpha$ EMCGG

Similarly, we know that $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$. If we let $\alpha = \beta$, then we have:

$$\begin{aligned} \cos(2\alpha) &= \cos(\alpha+\alpha) \\ &= \cos\alpha\cos\alpha - \sin\alpha\sin\alpha \\ \therefore \cos 2\alpha &= \cos^2\alpha - \sin^2\alpha \end{aligned}$$

Using the square identity, $\sin^2\alpha + \cos^2\alpha = 1$, we can also derive the following formulae:

$$\begin{aligned} \cos 2\alpha &= \cos^2\alpha - \sin^2\alpha \\ &= \left(1 - \sin^2\alpha\right) - \sin^2\alpha \\ \therefore \cos 2\alpha &= 1 - 2\sin^2\alpha \end{aligned}$$

And

$$\begin{aligned} \cos 2\alpha &= \cos^2\alpha - \sin^2\alpha \\ &= \cos^2\alpha - \left(1 - \cos^2\alpha\right) \\ &= \cos^2\alpha - 1 + \cos^2\alpha \\ \therefore \cos 2\alpha &= 2\cos^2\alpha - 1 \end{aligned}$$

Double angle formulae

- $\sin 2\alpha = 2\sin\alpha\cos\alpha$
- $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$
- $\cos 2\alpha = 1 - 2\sin^2\alpha$
- $\cos 2\alpha = 2\cos^2\alpha - 1$

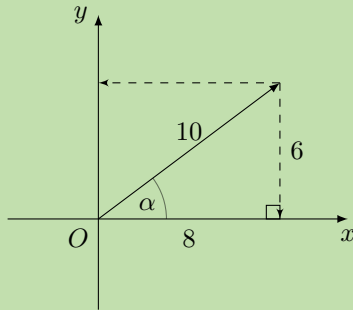
Worked example 7: Double angle identities

QUESTION

If α is an acute angle and $\sin\alpha = 0{,}6$, determine the value of $\sin 2\alpha$ without using a calculator.

SOLUTION

Step 1: Draw a sketch

We convert $0{,}6$ to a fraction so that we can use the ratio to represent the sides of a triangle.

$$\begin{aligned}\sin\alpha &= 0{,}6\\ &= \frac{6}{10}\end{aligned}$$

Using Pythagoras:

$$\begin{aligned}x^2 &= r^2 - y^2\\ &= 10^2 - 6^2\\ &= 100 - 36\\ &= 64\\ \therefore x &= 8\end{aligned}$$

Step 2: Use the double angle formula to determine the value of $\sin 2\alpha$

$$\begin{aligned}\sin 2\alpha &= 2\sin\alpha\cos\alpha\\ &= 2\left(\frac{6}{10}\right)\left(\frac{8}{10}\right)\\ &= \frac{96}{100}\\ &= 0{,}96\end{aligned}$$

Step 3: Write the final answer

$$\sin 2\alpha = 0{,}96$$

Check the answer using a calculator:

$$\begin{aligned}\sin\alpha &= 0{,}6\\ \therefore \alpha &\approx 36{,}87^\circ\\ 2\alpha &\approx 73{,}74^\circ\\ \therefore \sin(73{,}74^\circ) &\approx 0{,}96\end{aligned}$$

Worked example 8: Double angle identities

QUESTION

Prove that $\frac{\sin\theta+\sin 2\theta}{1+\cos\theta+\cos 2\theta} = \tan\theta$.

For which values of θ is the identity not valid?

SOLUTION

Step 1: Consider the given expressions

The right-hand side (RHS) of the identity cannot be simplified, so we simplify the left-hand side (LHS). We also notice that the trigonometric function on the RHS does not have a 2θ dependence, therefore we will need to use the double angle formulae to simplify $\sin 2\theta$ and $\cos 2\theta$ on the LHS.

Step 2: Prove the left-hand side equals the right-hand side

$$\begin{aligned}
\text{LHS} &= \frac{\sin\theta + 2\sin\theta\cos\theta}{1+\cos\theta+(2\cos^2\theta - 1)} \\
&= \frac{\sin\theta\,(1+2\cos\theta)}{\cos\theta\,(1+2\cos\theta)} \qquad \text{(factorise)} \\
&= \frac{\sin\theta}{\cos\theta} \\
&= \tan\theta \\
&= \text{RHS}
\end{aligned}$$

Step 3: Identify restricted values of θ

We know that $\tan\theta$ is undefined for $\theta = 90^\circ + k\ .\ 180^\circ, k \in \mathbb{Z}$.

Note that division by zero on the LHS is not allowed, so the identity will also be undefined for:

$$\begin{aligned}
1+\cos\theta+\cos 2\theta &= 0 \\
\cos\theta\,(1+2\cos\theta) &= 0 \\
\therefore \cos\theta = 0 &\text{ or } 1+2\cos\theta = 0
\end{aligned}$$

$$\text{For } \cos\theta = 0, \quad \theta = 90^\circ + k\ .\ 180^\circ$$

$$\begin{aligned}
\text{For } 1+2\cos\theta = 0, \quad \cos\theta &= -\frac{1}{2} \\
\therefore \theta &= 120^\circ + k\ .\ 360^\circ \\
\text{or } \theta &= 240^\circ + k\ .\ 360^\circ
\end{aligned}$$

for $k \in \mathbb{Z}$.

Exercise 4 – 3: Double angle identities

1. Given $5\cos\theta = -3$ and $\theta < 180°$. Determine the value of the following, without a calculator:

   a) $\cos 2\theta$
   b) $\sin(180° - 2\theta)$
   c) $\tan 2\theta$

2. Given $\cos 40° = t$, determine (without a calculator):

   a) $\cos 140°$
   b) $\sin 40°$
   c) $\sin 50°$
   d) $\cos 80°$
   e) $\cos 860°$
   f) $\cos(-1160°)$

3. a) Prove the identity: $\frac{1}{\sin 2A} - \frac{1}{\tan 2A} = \tan A$
   b) Hence, solve the equation $\frac{1}{\sin 2A} - \frac{1}{\tan 2A} = 0{,}75$ for $0° < A < 360°$.
4. Without using a calculator, find the value of the following:

   a) $\sin 22{,}5°$
   b) $\cos 67{,}5°$

5. a) Prove the identity: $\tan 2x + \frac{1}{\cos 2x} = \frac{\sin x + \cos x}{\cos x - \sin x}$
   b) Explain why the identity is undefined for $x = 45°$
6. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 28NJ 2. 28NK 3. 28NM 4a. 28NN 4b. 28NP 5. 28NQ

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4.4 Solving equations EMCGH

The general solution EMCGJ

The periodicity of the trigonometric functions means that there are an infinite number of positive and negative angles that satisfy an equation. If we do not restrict the solution, then we need to determine the general solution to the equation. We know that the sine and cosine functions have a period of $360°$ and the tangent function has a period of $180°$.

Method for finding the solution:

1. Simplify the equation using algebraic methods and trigonometric identities.
2. Determine the reference angle (use a positive value).
3. Use the CAST diagram to determine where the function is positive or negative (depending on the given equation/information).
4. Restricted values: find the angles that lie within a specified interval by adding/-subtracting multiples of the appropriate period.
5. General solution: find the angles in the interval $[0°; 360°]$ that satisfy the equation and add multiples of the period to each answer.
6. Check answers using a calculator.

General solutions:

1.

$$\begin{aligned} \text{If } \sin\theta &= x \\ \theta &= \sin^{-1} x + k \;.\; 360^\circ \\ \text{or } \theta &= \left(180^\circ - \sin^{-1} x\right) + k \;.\; 360^\circ \end{aligned}$$

2.

$$\begin{aligned} \text{If } \cos\theta &= x \\ \theta &= \cos^{-1} x + k \;.\; 360^\circ \\ \text{or } \theta &= \left(360^\circ - \cos^{-1} x\right) + k \;.\; 360^\circ \end{aligned}$$

3.

$$\begin{aligned} \text{If } \tan\theta &= x \\ \theta &= \tan^{-1} x + k \;.\; 180^\circ \end{aligned}$$

for $k \in \mathbb{Z}$.

Worked example 9: Finding the general solution

QUESTION

Determine the general solution for $\sin\theta = 0{,}3$ (correct to one decimal place).

SOLUTION

Step 1: Use a calculator to find the reference angle

$$\begin{aligned} \sin\theta &= 0{,}3 \\ \therefore \text{ ref } \angle &= \sin^{-1} 0{,}3 \\ &= 17{,}5^\circ \end{aligned}$$

Step 2: Use CAST diagram to determine in which quadrants $\sin\theta$ is positive

The CAST diagram indicates that $\sin\theta$ is positive in the first and second quadrants.

Using reduction formulae, we know that $\sin(180^\circ - \theta) = \sin\theta$.

In the first quadrant:

$$\begin{aligned} \theta &= 17{,}5^\circ \\ \therefore \theta &= 17{,}5^\circ + k \;.\; 360^\circ \end{aligned}$$

In the second quadrant:

$$\begin{aligned} \theta &= 180^\circ - 17{,}5^\circ \\ \therefore \theta &= 162{,}5^\circ + k \;.\; 360^\circ \end{aligned}$$

where $k \in \mathbb{Z}$.

Step 3: Check that the solution satisfies the original equation

We can select random values of k to check that the answers satisfy the original equation.

Let $k = 4$:

$$\begin{aligned} \theta &= 17{,}5^\circ + 4(360)^\circ \\ \therefore \theta &= 1457{,}5^\circ \\ \text{And } \sin 1457{,}5^\circ &= 0{,}3007\ldots \end{aligned}$$

This solution is correct.

Similarly, if we let $k = -2$:

$$\begin{aligned} \theta &= 162{,}5^\circ - 2(360)^\circ \\ \therefore \theta &= -557{,}5^\circ \\ \text{And } \sin(-557{,}5^\circ) &= 0{,}3007\ldots \end{aligned}$$

This solution is also correct.

Step 4: Write the final answer

$\theta = 17{,}5^\circ + k\ .\ 360^\circ$ or $\theta = 162{,}5^\circ + k\ .\ 360^\circ$ for $k \in \mathbb{Z}$.

Worked example 10: Trigonometric equations

QUESTION

Solve the following equation for y, without using a calculator:

$\frac{1-\sin y - \cos 2y}{\sin 2y - \cos y} = -1$

SOLUTION

Step 1: Simplify the equation

We first simplify the left-hand side of the equation using the double angle formulae. To solve this equation, we need to manipulate the given equation to be of the form:

$$\text{single trigonometric ratio} = \text{constant}$$

$$\begin{aligned} \frac{1 - \sin y - \left(1 - 2\sin^2 y\right)}{2\sin y \cos y - \cos y} &= -1 \\ \frac{2\sin^2 y - \sin y}{\cos y\,(2\sin y - 1)} &= -1 \\ \frac{\sin y\,(2\sin y - 1)}{\cos y\,(2\sin y - 1)} &= -1 \\ \frac{\sin y}{\cos y} &= -1 \\ \therefore \tan y &= -1 \end{aligned}$$

Step 2: Use a calculator to find the reference angle

$$\begin{aligned} \tan y &= -1 \\ \therefore \text{ref } \angle &= \tan^{-1}(1) \\ &= 45° \end{aligned}$$

Step 3: Use CAST diagram to determine in which quadrants $\tan y$ is negative

The CAST diagram indicates that $\tan y$ is negative in the second and fourth quadrants.

$$\begin{aligned} y &= (180° - 45°) + k \,.\, 180° \\ \therefore y &= 135° + k \,.\, 180°, \quad k \in \mathbb{Z} \end{aligned}$$

Notice that for $k = 1$, $y = 135° + 180° = 315°$, which is the angle in the fourth quadrant.

Step 4: Check that the solution satisfies the original equation

Step 5: Write the final answer

$y = 135° + k \,.\, 180°$ where $k \in \mathbb{Z}$.

Worked example 11: Trigonometric equations

QUESTION

Prove $8\cos^4 x - 8\cos^2 x + 1 = \cos 4x$ and hence solve $8\cos^4 x - 8\cos^2 x + 1 = 0{,}8$ (correct to one decimal place).

SOLUTION

Step 1: Prove the identity

Expand the right-hand side of the identity and show that it is equal to the left-hand side:

$$\begin{aligned} \text{RHS} &= \cos 4x \\ &= 2\cos^2 2x - 1 \\ &= 2\left(2\cos^2 x - 1\right)^2 - 1 \\ &= 2\left(4\cos^4 x - 4\cos^2 x + 1\right) - 1 \\ &= 8\cos^4 x - 8\cos^2 x + 2 - 1 \\ &= 8\cos^4 x - 8\cos^2 x + 1 \\ &= \text{LHS} \end{aligned}$$

Step 2: Solve the equation

$$\begin{aligned} 8\cos^4 x - 8\cos^2 x + 1 &= 0{,}8 \\ \therefore \cos 4x &= 0{,}8 \end{aligned}$$

Step 3: Use a calculator to find the reference angle

$$\begin{aligned} \cos 4x &= 0{,}8 \\ \therefore \text{ref } \angle &= \cos^{-1}(0{,}8) \\ &= 36{,}9^\circ \end{aligned}$$

Step 4: Use CAST diagram to determine in which quadrants $\cos 4x$ is positive

The CAST diagram indicates that $\cos 4x$ is positive in the first and fourth quadrants.

In the first quadrant:

$$\begin{aligned} 4x &= 36{,}9^\circ + k \,.\, 360^\circ \\ \therefore x &= 9{,}2^\circ + k \,.\, 90^\circ \end{aligned}$$

Important: remember to also divide $k \,.\, 360^\circ$ by 4.

In the fourth quadrant:

$$\begin{aligned} 4x &= (360^\circ - 36{,}9^\circ) + k \,.\, 360^\circ \\ &= 323{,}1^\circ + k \,.\, 360^\circ \\ \therefore x &= 80{,}8^\circ + k \,.\, 90^\circ \end{aligned}$$

where $k \in \mathbb{Z}$.

Step 5: Check that the solution satisfies the original equation

Step 6: Write the final answer

$$\begin{aligned} x &= 9{,}2^\circ + k \,.\, 90^\circ \\ \text{or } x &= 80{,}8^\circ + k \,.\, 90^\circ, \quad k \in \mathbb{Z} \end{aligned}$$

Worked example 12: Trigonometric equations

QUESTION

Find the general solution for $\sin\theta\cos^2\theta = \sin^3\theta$.

SOLUTION

Step 1: Simplify the given equation

Do not divide both sides of the equation by $\sin\theta$:

- part of the solution would be lost;
- we would need to restrict the values of θ to those where $\sin\theta \neq 0$ (division by zero is not permitted).

$$\begin{aligned}
\sin\theta\cos^2\theta &= \sin^3\theta \\
\sin\theta\cos^2\theta - \sin^3\theta &= 0 \\
\sin\theta\left(\cos^2\theta - \sin^2\theta\right) &= 0 \\
\sin\theta\left(\cos\theta - \sin\theta\right)\left(\cos\theta + \sin\theta\right) &= 0
\end{aligned}$$

Step 2: Apply the zero product law and solve for θ

$$\sin\theta\left(\cos\theta - \sin\theta\right)\left(\cos\theta + \sin\theta\right) = 0$$

$$\begin{aligned}
&\sin\theta = 0 \\
&\therefore \theta = 0^\circ + k\ .\ 360^\circ \\
&\text{or } \theta = 180^\circ + k\ .\ 360^\circ, \quad k \in \mathbb{Z}
\end{aligned}$$

$$\begin{aligned}
\cos\theta - \sin\theta &= 0 \\
\cos\theta &= \sin\theta \\
\cos\theta &= \cos\left(90^\circ - \theta\right) \\
\therefore \theta &= \left(90^\circ - \theta\right) + k\ .\ 360^\circ \\
2\theta &= 90^\circ + k\ .\ 360^\circ \\
\therefore \theta &= 45^\circ + k\ .\ 180^\circ, \quad k \in \mathbb{Z}
\end{aligned}$$

$$\begin{aligned}
\cos\theta + \sin\theta &= 0 \\
\cos\theta &= -\sin\theta \\
\sin\left(90^\circ + \theta\right) &= \sin(-\theta) \\
\therefore 90^\circ + \theta &= -\theta + k\ .\ 360^\circ \\
2\theta &= -90^\circ + k\ .\ 360^\circ \\
\therefore \theta &= -45^\circ + k\ .\ 180^\circ, \quad k \in \mathbb{Z}
\end{aligned}$$

Step 3: Write the final answer

$$\begin{aligned}
&\theta = 0^\circ + k\ .\ 360^\circ \\
&\text{or } \theta = 180^\circ + k\ .\ 360^\circ \\
&\text{or } \theta = \pm 45^\circ + k\ .\ 180^\circ, \quad k \in \mathbb{Z}
\end{aligned}$$

Exercise 4 – 4: Solving trigonometric equations

1. Find the general solution for each of the following equations (correct to two decimal places):

   a) $\sin 2x = \tan 28°$
   b) $\cos y = \sin 2y$
   c) $\sin 2\alpha = \cos 2\alpha$
   d) $\sin 3p = \sin 2p$
   e) $\tan A = \frac{1}{\tan A}$
   f) $\sin x \tan x = 1$
   g) $\sin t \ . \ \sin 2t + \cos 2t = 1$
   h) $\sin 60° \cos x + \cos 60° \sin x = 1$

2. Given: $\sin x \cos x = \sqrt{3} \sin^2 x$

   a) Solve the equation for $x \in [0°; 360°]$, without using a calculator.
   b) Draw a graph and indicate the solution on the diagram.

3. Given: $1 + \tan^2 2A = 5 \tan 2A - 5$

   a) Determine the general solution.
   b) How many solutions does the given equation have in the interval $[-90°; 360°]$?

4. Without using a calculator, solve $\cos(A - 25°) + \cos(A + 25°) = \cos 25°$ in $[-360°; 360°]$.

5. a) Find the general solution for $\sin x \cos 3x + \cos x \sin 3x = \tan 140°$.
   b) Use a graph to illustrate the solution for the interval $[0°; 90°]$.

6. Explain why the general solution for the equation $\cos\theta = a$ is $\theta = \cos^{-1} a + k \ . \ 360°$ and the general solution for $\tan\theta = a$ is $\theta = \tan^{-1} a + k \ . \ 180°$. Why are they different?

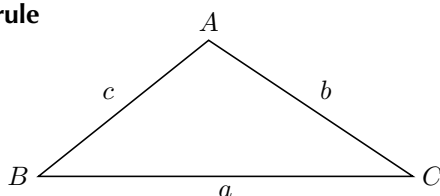
7. Solve for x: $\sqrt{3} \sin x + \cos x = 2$

8. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 28NR 1b. 28NS 1c. 28NT 1d. 28NV 1e. 28NW 1f. 28NX
1g. 28NY 1h. 28NZ 2. 28P2 3. 28P3 4. 28P4 5. 28P5
6. 28P6 7. 28P7

4.5 Applications of trigonometric functions EMCGK

Area, sine and cosine rule

Area rule	Sine rule	Cosine rule
area $\triangle ABC = \frac{1}{2}bc\sin\hat{A}$	$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin\hat{C}}{c}$	$a^2 = b^2 + c^2 - 2bc\cos A$
area $\triangle ABC = \frac{1}{2}ac\sin\hat{B}$	$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin\hat{C}}$	$b^2 = a^2 + c^2 - 2ac\cos B$
area $\triangle ABC = \frac{1}{2}ab\sin\hat{C}$		$c^2 = a^2 + b^2 - 2ab\cos\hat{C}$

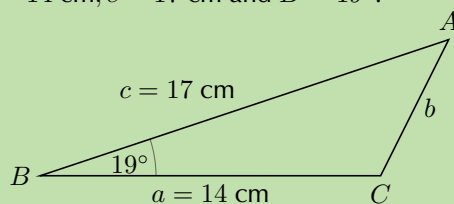
How to determine which rule to use:

1. Area rule:
    - no perpendicular height is given
2. Sine rule:
    - no right angle is given
    - two sides and an angle are given (**not** the included angle)
    - two angles and a side are given
3. Cosine rule:
    - no right angle is given
    - two sides and the included angle are given
    - three sides are given

Problems in two dimensions EMCGM

Worked example 13: Area, sine and cosine rule

QUESTION

Given $\triangle ABC$ with $a = 14$ cm, $c = 17$ cm and $\hat{B} = 19°$.

Calculate the following:

1. b
2. $\hat{C}$
3. Area $\triangle ABC$

SOLUTION

Step 1: Use the cosine rule to determine the length of b

$$\begin{aligned} b^2 &= a^2 + c^2 - 2ac\cos\hat{B} \\ &= (14)^2 + (17)^2 - 2(14)(17)\cos 19° \\ &= 34{,}93\ldots \\ \therefore b &= 5{,}9 \text{ cm} \end{aligned}$$

Step 2: Use the sine rule to determine $\hat{C}$

$$\begin{aligned} \frac{\sin\hat{C}}{c} &= \frac{\sin\hat{B}}{b} \\ \frac{\sin\hat{C}}{17} &= \frac{\sin 19°}{5{,}9} \\ \sin\hat{C} &= \frac{17\times\sin 19°}{5{,}9} \\ \therefore \sin\hat{C} &= 0{,}938\ldots \end{aligned}$$

$$\begin{aligned} \text{First quadrant: } \hat{C} &= 69{,}7° \\ \text{Second quadrant: } \hat{C} &= 180° - 69{,}7° \\ &= 110{,}3° \end{aligned}$$

From the diagram, we see that $\hat{C} > 90°$, therefore $\hat{C} = 110{,}3°$.

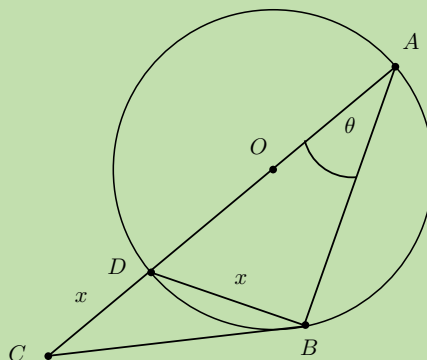
Step 3: Calculate the area $\triangle ABC$

$$\begin{aligned} \text{Area } \triangle ABC &= \frac{1}{2}ac\sin\hat{B} \\ &= \frac{1}{2}(14)(17)\sin 19° \\ &= 38{,}7 \text{ cm}^2 \end{aligned}$$

Worked example 14: Problem in two dimensions

QUESTION

In the figure below, $CD = BD = x$ and $B\hat{A}D = \theta$.

Show that $BC^2 = 2x^2(1 + \sin\theta)$.

SOLUTION

Step 1: Consider the given information

Use the given information to determine as many of the unknown angles as possible.

$$CD = BD = x \quad \text{(given)}$$

$$B\hat{A}D = \theta \quad \text{(given)}$$

$$D\hat{B}A = 90^\circ \quad (\angle \text{ in semi circle})$$

$$B\hat{D}A = 180^\circ - 90^\circ - \theta \quad (\angle\text{s sum of } \triangle ABD)$$

$$= 90^\circ - \theta$$

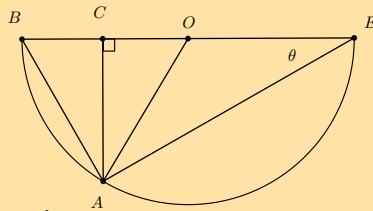
$$B\hat{D}C = 90^\circ + \theta \quad (\angle\text{s on a str. line})$$

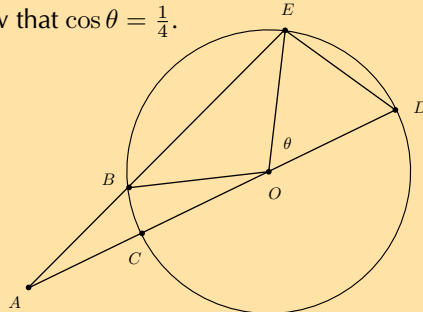
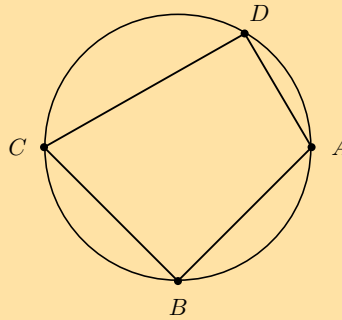
Step 2: Determine the expression for BC

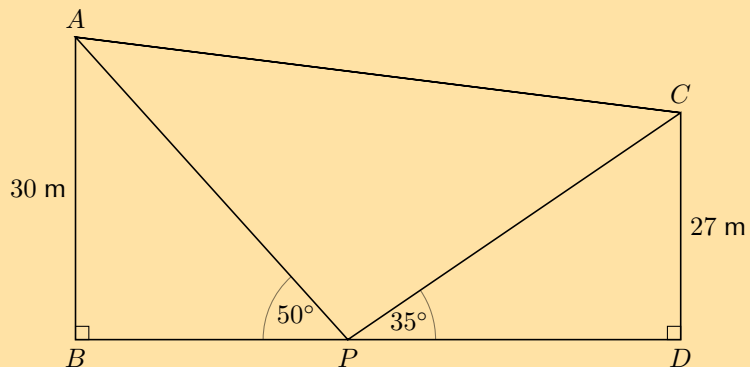
To derive the required expression, we need to write BC in terms of x and θ. In $\triangle CDB$, we can use the cosine rule to determine BC:

$$\begin{aligned} BC^2 &= CD^2 + BD^2 - 2\ .\ CD\ .\ BD\ .\ \cos\left(B\hat{D}C\right) \\ &= x^2 + x^2 - 2x^2 \cos(90^\circ + \theta) \\ &= 2x^2 - 2x^2(-\sin\theta) \\ &= 2x^2 + 2x^2 \sin\theta \\ &= 2x^2(1 + \sin\theta) \end{aligned}$$

Exercise 4 – 5: Problems in two dimensions

1. In the diagram below, O is the centre of the semi-circle BAE.

   a) Find $A\hat{O}C$ in terms of θ.
   b) In $\triangle ABE$, determine an expression for $\cos\theta$.
   c) In $\triangle ACE$, determine an expression for $\sin\theta$.
   d) In $\triangle ACO$, determine an expression for $\sin 2\theta$.
   e) Use the results from the previous questions to show that $\sin 2\theta = 2\sin\theta\cos\theta$.

2. DC is a diameter of the circle with centre O and radius r. $CA = r$, $AE = 2DE$ and $D\hat{O}E = \theta$. Show that $\cos\theta = \frac{1}{4}$.

3. The figure below shows a cyclic quadrilateral with $\frac{BC}{CD} = \frac{AD}{AB}$.

   a) Show that the area of the cyclic quadrilateral is $DC \cdot DA \cdot \sin \hat{D}$.
   b) Write down two expressions for CA^2: one in terms of $\cos \hat{D}$ and one in terms of $\cos \hat{B}$.
   c) Show that $2CA^2 = CD^2 + DA^2 + AB^2 + BC^2$.
   d) Suppose that $BC = 10$ units, $CD = 15$ units, $AD = 4$ units and $AB = 6$ units. Calculate CA^2 (correct to one decimal place).
   e) Find the angle $\hat{B}$. Hence, calculate the area of $ABCD$ (correct to one decimal place).

4. Two vertical towers AB and CD are 30 m and 27 m high, respectively. Point P lies between the two towers. The angle of elevation from P to A is $50°$ and from P to C is $35°$. A cable is needed to connect A and C.

   a) Determine the minimum length of cable needed to connect A and C (to the nearest metre).
   b) How far apart are the bases of the two towers (to the nearest metre)?

5. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 28P8 1b. 28P9 1c. 28PB 1d. 28PC 1e. 28PD 2. 28PF

3a. 28PG 3b. 28PH 3c. 28PJ 3d. 28PK 3e. 28PM 4. 28PN

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Trigonometric formulae are useful for solving problems in two dimensions. However, in the real world all objects are three dimensional, so it is important that we extend the application of the area, sine and cosine formulae to three dimensional situations.

Drawing a three dimensional diagram is a crucial step in finding the solution to a problem. Interpreting the given information and sketching a three dimensional diagram are skills that need to be practised.

Worked example 15: Problems in three dimensions - height of a pole

QUESTION

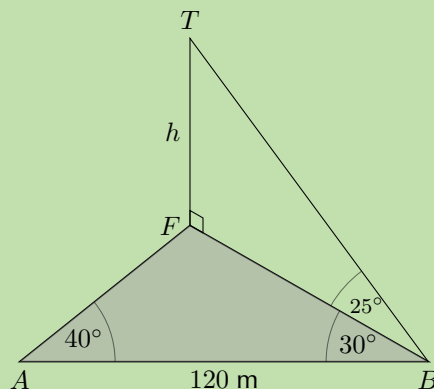
T is the top of a pole and its base, F, is in the same horizontal plane as the points A and B. The angle of elevation measured from B to T is $25°$. $AB = 120$ m, $F\hat{A}B = 40°$ and $F\hat{B}A = 30°$.

Use the given information to calculate h, the height of the pole, to the nearest metre.

SOLUTION

Step 1: Draw a sketch

It is a challenge to analyze this situation when only a description is given. A diagram is very useful for representing three dimensional problems.

Draw a sketch using the given information. Indicate all right angles on the sketch, even those that do not look as though they are a $90°$ angle (for example, $T\hat{F}B = 90°$). It is also helpful to shade the horizontal plane ($\triangle FAB$), as it adds depth to the diagram and gives a better visualisation of the situation.

Step 2: Consider the given information

There are two triangles to consider: $\triangle FAB$ and $\triangle TFB$. We are required to find the length of FT, but the only information we have for $\triangle TFB$ is $F\hat{B}T = 25°$.

Important: notice that FB is a side of $\triangle TFB$ and it is also a side of $\triangle FAB$ - it is a link or bridge between the two triangles.

Therefore, to determine the height of the pole:

1. Write down all the given information on the sketch.
2. Use the information for $\triangle FAB$ to calculate FB.
3. Use FB and the information for $\triangle TFB$ to calculate FT.

Step 3: Determine the length of FB

We notice that $\triangle FAB$ does not have a right-angle, so we use the sine rule to determine FB.

$$\begin{aligned} \text{In } \triangle FAB: \quad \hat{F} &= 180° - 40° - 30° \qquad (\angle\text{s sum of } \triangle FAB) \\ &= 110° \end{aligned}$$

$$\begin{aligned} \frac{FB}{\sin \hat{A}} &= \frac{AB}{\sin \hat{F}} \\ \frac{FB}{\sin 40°} &= \frac{120}{\sin 110°} \\ FB &= \frac{120 \times \sin 40°}{\sin 110°} \\ \therefore FB &= 82{,}084\ldots \end{aligned}$$

Step 4: Determine the length of FT

In $\triangle TFB$:

$$\begin{aligned} \hat{F} &= 90° && \text{(vertical pole)} \\ \hat{B} &= 25° && \text{(given)} \\ \tan \hat{B} &= \tfrac{FT}{FB} \\ \tan \hat{B} \times FB &= FT \\ \tan 25° \times 82{,}084\ldots &= FT \\ \therefore FT &= 38{,}246\ldots \\ &\approx 38 \text{ m} && \text{(to nearest metre)} \end{aligned}$$

Note: do not round off answers in the intermediate steps as this will affect the accuracy of the final answer. Always try to do the calculation in one complete step and only round off the final answer.

Step 5: Write the final answer

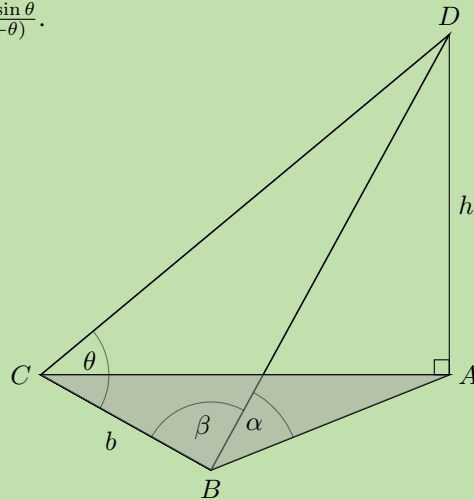
The height of the pole, h, is 38 m.

The calculation in the above worked example is only applicable for the specific numbers given. However, if we derived a general formula for this contextual situation, then it could be applied to any suitable set of numerical values.

Worked example 16: Problems in three dimensions - height of a building

QUESTION

D is the top of a building of height h. The base of the building is at A and $\triangle ABC$ lies on the ground (a horizontal plane). $BC = b$, $D\hat{B}A = \alpha$, $D\hat{B}C = \beta$ and $D\hat{C}B = \theta$.

Show that $h = \frac{b \sin\alpha \sin\theta}{\sin(\beta+\theta)}$.

SOLUTION

Step 1: Consider the given information

We know that $\triangle ABD$ is right-angled and we are required to find a formula to calculate the length of AD. In $\triangle BCD$, we are given two angles and a length of BC. We identify the side BD as the link between $\triangle ABD$ and $\triangle BCD$.

Step 2: Determine an expression for BD

$\triangle BCD$ does not have a right-angle but two angles and a side are given, so we use the sine rule to determine BD.

$$\text{In } \triangle BCD: \quad B\hat{D}C = 180° - (\beta + \theta) \qquad (\angle\text{s sum of } \triangle BCD)$$

$$\therefore \sin(180° - (\beta+\theta)) = \sin(\beta+\theta)$$

$$\frac{BD}{\sin\theta} = \frac{b}{\sin\left(B\hat{D}C\right)}$$

$$BD = \frac{b\sin\theta}{\sin\left(B\hat{D}C\right)}$$

$$= \frac{b\sin\theta}{\sin(\beta+\theta)}$$

Step 3: Determine an expression for AD

In $\triangle ABD$:

$$B\hat{A}D = 90° \qquad \text{(building vertical)}$$

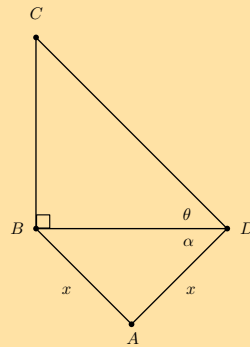
$$D\hat{B}A = \alpha \qquad \text{(given)}$$

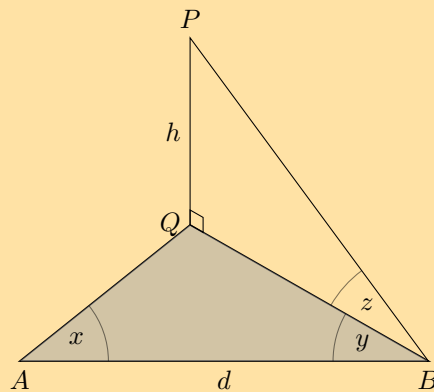
$$\frac{h}{BD} = \sin\alpha$$

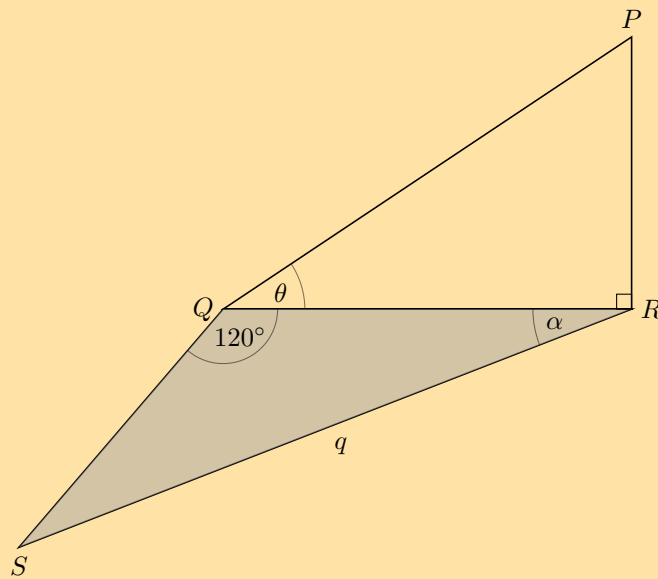
$$h = BD\sin\alpha$$

$$\therefore h = \frac{b\sin\alpha\sin\theta}{\sin(\beta+\theta)}$$

Exercise 4 – 6: Problems in three dimensions

1. The line BC represents a tall tower, with B at its base. The angle of elevation from D to C is θ. A man stands at A such that $BA = AD = x$ and $A\hat{D}B = \alpha$.

   a) Find the height of the tower BC in terms of x, $\tan\theta$ and $\cos\alpha$.
   b) Find BC if we are given that $x = 140$ m, $\alpha = 21°$ and $\theta = 9°$.

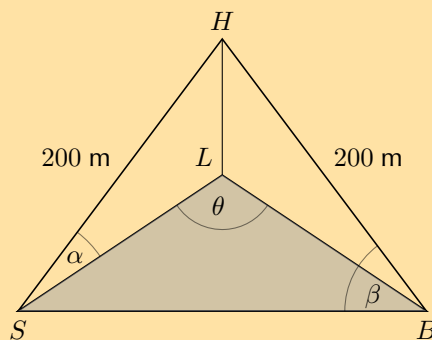
2. P is the top of a mast and its base, Q, is in the same horizontal plane as the points A and B. The angle of elevation measured from B to P is z. $AB = d$, $Q\hat{A}B = x$ and $Q\hat{B}A = y$.

   a) Use the given information to derive a general formula for h, the height of the mast.
   b) If $d = 50$ m, $x = 46°$, $y = 15°$ and $z = 20°$, calculate h (to the nearest metre).

3. PR is the height of a block of flats with R at the base and P at the top of the building. S is a point in the same horizontal plane as points Q and R. $SR = q$ units, $S\hat{Q}R = 120°$, $S\hat{R}Q = \alpha$ and $R\hat{Q}P = \theta$.

a) Show that the height of the block of flats, PR, can be expressed as:

$$PR = q \tan\theta \left(\cos\alpha - \frac{\sqrt{3}\sin\alpha}{3} \right)$$

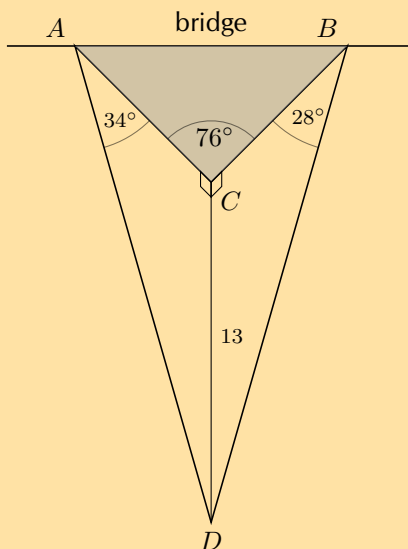
b) If $SR = 35$ m, $S\hat{R}Q = 16°$ and $R\hat{Q}P = 30°$, calculate PR (correct to one decimal place).

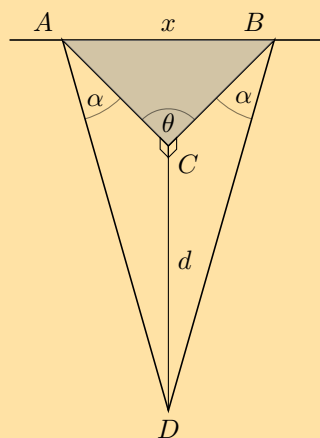
c) Assuming each level is 2,5 m high, estimate the number of levels in the block of flats.

4. Two ships at sea can see a lighthouse on the shore. The distance from the top of the lighthouse (H) to ship S and to ship B is 200 m. The angle of elevation from S to H is α, $H\hat{B}S = \beta$ and $S\hat{L}B = \theta$

a) Show that the distance between the two ships is given by $SB = 400\cos\beta$.

b) Show that the area of the sea included in $\triangle LSB$ is given by area $\triangle LSB = 2000\cos^2\alpha\sin\theta$.

c) Calculate the triangular area of the sea if the angle of inclination from the ship to the top of the lighthouse is $10°$ and the angle between the direct lines from the base of the lighthouse to each ship is $85°$.

5. A triangular look-out platform ($\triangle ABC$) is attached to a bridge that extends over a deep gorge. The vertical depth of the gorge, the distance from the edge of the look-out C to the bottom of the gorge D, is 13 m. The angle of depression from A to D is $34°$ and from B to D is $28°$. The angle at the edge of the platform, $\hat{C}$ is $76°$.

a) Calculate the area of the look-out platform (to the nearest m^2).

b) If the platform is constructed so that the two angles of depression, $C\hat{A}D$ and $C\hat{B}D$, are both equal to $45°$ and the vertical depth of the gorge $CD = d$, $AB = x$ and $A\hat{C}B = \theta$, show that $\cos\theta = 1 - \frac{x^2}{2d^2}$.

c) If $AB = 25$ m and $CD = 13$ m, calculate $A\hat{C}B$ (to the nearest integer).

6. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 28PP 2. 28PQ 3. 28PR 4. 28PS 5. 28PT

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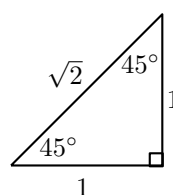
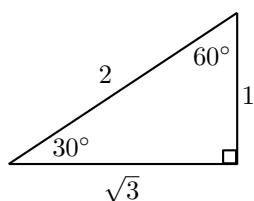
4.6 Summary EMCGP

Pythagorean Identities	Ratio Identities
$\cos^2\theta + \sin^2\theta = 1$	$\tan\theta = \frac{\sin\theta}{\cos\theta}$
$\cos^2\theta = 1 - \sin^2\theta$	$\frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta}$
$\sin^2\theta = 1 - \cos^2\theta$	

Special angle triangles

θ	$0°$	$30°$	$45°$	$60°$	$90°$
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undef

CAST diagram and reduction formulae

y

Second Quadrant:

- **sine function is positive**
- $\sin(180° - \theta) = \sin\theta$
- $\cos(180° - \theta) = -\cos\theta$
- $\tan(180° - \theta) = -\tan\theta$
- $\sin(90° + \theta) = \cos\theta$
- $\cos(90° + \theta) = -\sin\theta$

First Quadrant:

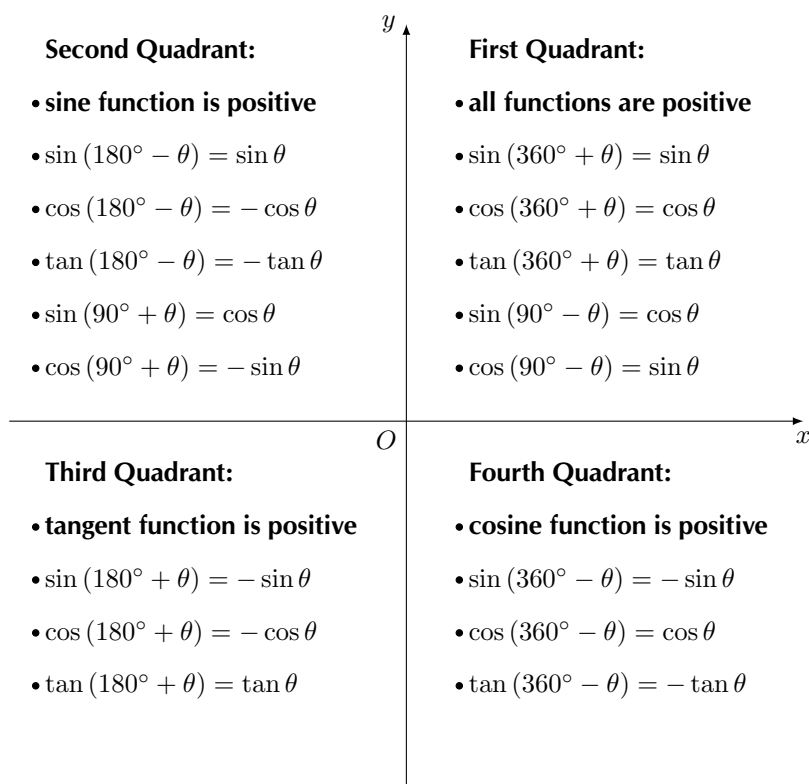
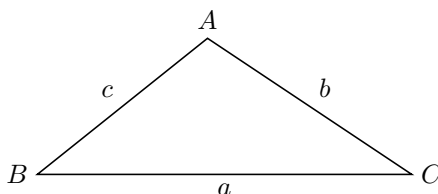
- **all functions are positive**
- $\sin(360° + \theta) = \sin\theta$
- $\cos(360° + \theta) = \cos\theta$
- $\tan(360° + \theta) = \tan\theta$
- $\sin(90° - \theta) = \cos\theta$
- $\cos(90° - \theta) = \sin\theta$

O x

Third Quadrant:

- **tangent function is positive**
- $\sin(180° + \theta) = -\sin\theta$
- $\cos(180° + \theta) = -\cos\theta$
- $\tan(180° + \theta) = \tan\theta$

Fourth Quadrant:

- **cosine function is positive**
- $\sin(360° - \theta) = -\sin\theta$
- $\cos(360° - \theta) = \cos\theta$
- $\tan(360° - \theta) = -\tan\theta$

Negative angles	Periodicity Identities	Cofunction Identities
$\sin(-\theta) = -\sin\theta$	$\sin\left(\theta \pm 360^{\circ}\right) = \sin\theta$	$\sin\left(90^{\circ} - \theta\right) = \cos\theta$
$\cos(-\theta) = \cos\theta$	$\cos\left(\theta \pm 360^{\circ}\right) = \cos\theta$	$\cos\left(90^{\circ} - \theta\right) = \sin\theta$
$\tan(-\theta) = -\tan\theta$	$\tan\left(\theta \pm 180^{\circ}\right) = \tan\theta$	$\sin\left(90^{\circ} + \theta\right) = \cos\theta$
		$\cos\left(90^{\circ} + \theta\right) = -\sin\theta$

Area Rule	Sine Rule	Cosine Rule
Area $= \frac{1}{2}bc\sin\hat{A}$	$\frac{\sin\hat{A}}{a} = \frac{\sin\hat{B}}{b} = \frac{\sin\hat{C}}{c}$	$a^2 = b^2 + c^2 - 2bc\cos\hat{A}$
Area $= \frac{1}{2}ab\sin\hat{C}$	$a\sin\hat{B} = b\sin\hat{A}$	$b^2 = a^2 + c^2 - 2ac\cos\hat{B}$
Area $= \frac{1}{2}ac\sin\hat{B}$	$b\sin C = c\sin\hat{B}$	$c^2 = a^2 + b^2 - 2ab\cos\hat{C}$
	$a\sin C = c\sin\hat{A}$	

Compound Angle Identities	Double Angle Identities
$\sin(\theta + \beta) = \sin\theta\cos\beta + \cos\theta\sin\beta$	$\sin(2\theta) = 2\sin\theta\cos\theta$
$\sin(\theta - \beta) = \sin\theta\cos\beta - \cos\theta\sin\beta$	$\cos(2\theta) = \cos^2\theta - \sin^2\theta$
$\cos(\theta + \beta) = \cos\theta\cos\beta - \sin\theta\sin\beta$	$\cos(2\theta) = 1 - 2\sin^2\theta$
$\cos(\theta - \beta) = \cos\theta\cos\beta + \sin\theta\sin\beta$	$\cos(2\theta) = 2\cos^2\theta - 1$
	$\tan(2\theta) = \frac{\sin 2\theta}{\cos 2\theta}$

See video: 28N2 at www.everythingmaths.co.za

Exercise 4 – 7: End of chapter exercises

1. Determine the following without using a calculator:

   a) $\cos 15°$
   b) $\cos 75°$
   c) $\tan 75°$
   d) $\cos 3° \cos 42° - \sin 3° \sin 42°$
   e) $1 - 2\sin^2(22{,}5°)$

2. Given $\cos\theta = 0{,}7$. Using a diagram, find $\cos 2\theta$ and $\cos 4\theta$.
3. Given $7\sin\alpha = 3$ for $\alpha > 90°$.
   Determine the following (leave answers in surd form):

   a) $\cos 2\alpha$
   b) $\tan 2\alpha$

4. If $4\tan A + 3 = 0$ for $A < 270°$, determine, without the use of a calculator:

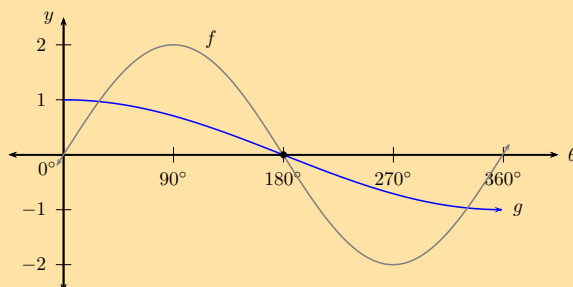
$$\left(\sin\frac{A}{2} - \cos\frac{A}{2}\right)\left(\sin\frac{A}{2} + \cos\frac{A}{2}\right)$$

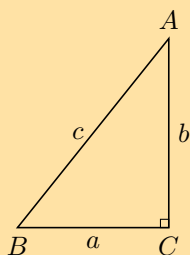
5. Simplify: $\cos 67° \cos 7° + \cos 23° \cos 83°$
6. Solve the equation:
   $\cos 3\theta \cos\theta - \sin 3\theta \sin\theta = -\frac{1}{2}$ for $\theta \in [-90°; 90°]$.
7. Find the general solution, without a calculator, for the following equations:
   a) $3\sin\theta = 2\cos^2\theta$
   b) $2\sin 2x - 2\cos x = \sqrt{2} - 2\sqrt{2}\sin x$
   c) $\cos x \cos 10° + \sin x \cos 100° = 1 - 2\sin^2 x$
   d) $6\sin^2\alpha + 2\sin 2\alpha - 1 = 0$
8. a) Prove: $\sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}$
   b) Hence, solve the equation $3\sin\theta - \sin 3\theta = 2$ for $\theta \in [0°; 360°]$.
9. Prove the following identities:
   a) $\cos^2\alpha\left(1 - \tan^2\alpha\right) = \cos 2\alpha$
   b) $4\sin\theta\cos\theta\cos 2\theta = \sin 4\theta$
   c) $4\cos^3 x - 3\cos x = \cos 3x$
   d) $\cos 2A + 2\sin 2A + 2 = (3\cos A + \sin A)(\cos A + \sin A)$
   e) $\dfrac{\cos 2x}{(\cos x + \sin x)^3} = \dfrac{\cos x - \sin x}{1 + \sin 2x}$
10. a) Prove: $\tan y = \dfrac{\sin 2y}{\cos 2y + 1}$
    b) For which values of y is the identity undefined?
11. Given: $1 + \tan^2 3\theta - 3\tan 3\theta = 5$
    a) Find the general solution.
    b) Find the solution for $\theta \in [0°; 90°]$.
    c) Draw a graph of $y = \tan 3\theta$ for $\theta \in [0°; 90°]$ and indicate the solutions to the equation on the graph.
    d) Use the graph to determine where $\tan 3\theta < -1$.
12. a) Show that:

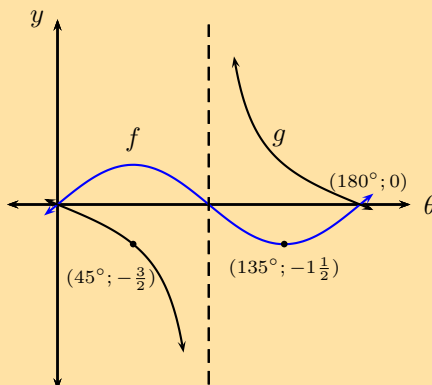
$$\sin(A + B) - \sin(A - B) = 2\cos A \sin B$$

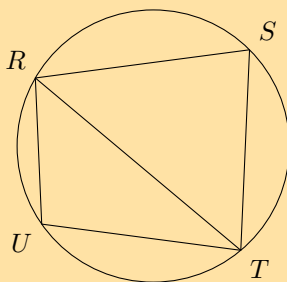
    b) Use this result to solve $\sin 3x - \sin x = 0$ for $x \in [-180°; 360°]$.
    c) On the same system of axes, draw two graphs to solve graphically: $\sin 3x - \sin x = 0$ for $x \in [0°; 360°]$. Indicate the solutions on the graph using the letters $A, B, \ldots$ etc.
13. Given: $\cos 2x = \sin x$ for $x \in [0°; 360°]$
    a) Solve for x algebraically.
    b) Verify the solution graphically by sketching two graphs on the same system of axes.

14. The following graphs are given below:

$$f : = a \sin x$$
$$g : = \cos bx \qquad (x \in [0°; 360°])$$

a) Explain why $a = 2$ and $b = \frac{1}{2}$.
b) For how many x-values in $[0°; 360°]$ will $f(x) - g(x) = 0$?
c) Use the graph to solve $f(x) - g(x) = 1$.
d) Solve $a \sin x = \cos bx$ for $x \in [0°; 360°]$ using trigonometric identities.
e) For which values of x will $\frac{1}{2} \cos\left(\frac{x}{2}\right) \leq \sin x$ for $x \in [0°; 360°]$?

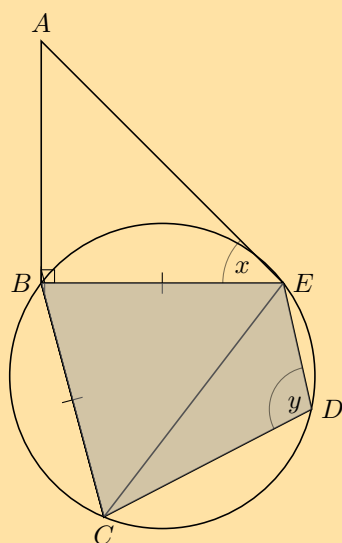
15. In $\triangle ABC$, $AB = c, BC = a, CA = b$ and $\hat{C} = 90°$.

a) Prove that $\sin 2A = \frac{2ab}{c^2}$.
b) Show that $\cos 2A = \frac{b^2 - a^2}{c^2}$.

16. Given the graphs of $f(\theta) = p \sin k\theta$ and $g(\theta) = q \tan \theta$, determine the values of p, k and q.

17. $\triangle RST$ is an acute angled triangle with $RS = ST = t$. Show that area $\triangle RST = t^2 \sin \hat{T} \cos \hat{T}$.

18. $RSTU$ is a cyclic quadrilateral with $RU = 6$ cm, $UT = 7{,}5$ cm, $RT = 11$ cm and $RS = 9{,}5$ cm.

a) Calculate $\hat{U}$.

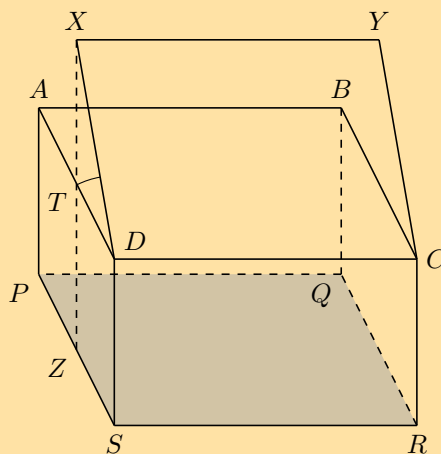
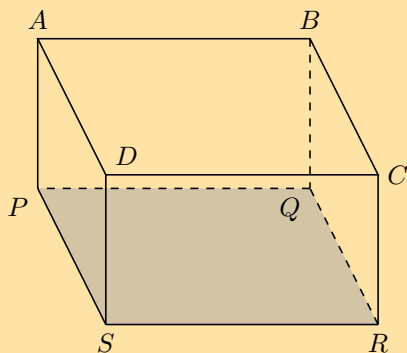
b) Determine $\hat{S}$.

c) Find $R\hat{T}S$.

19. $BCDE$ is a cyclic quadrilateral that lies in a horizontal plane. AB is a vertical pole with base B. The angle of elevation from E to A is $x°$ and $C\hat{D}E = y°$. $\triangle BEC$ is an isosceles triangle with $BE = BC$.

a) Show that $B\hat{C}E = \frac{1}{2}y$.

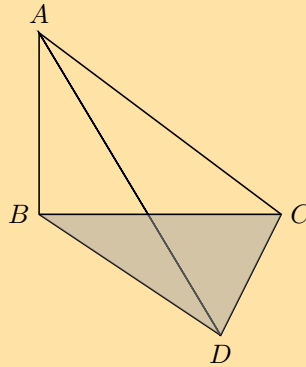
b) Show that $CE = 2BE\cos\left(\frac{y}{2}\right)$

c) If $AB = 2{,}6$ m, $x = 37°$ and $y = 109°$, calculate the length of CE.

20. The first diagram shows a rectangular box with $SR = 8$ cm, $PS = 6$ cm and $PA = 4$ cm. The lid of the box, $ABCD$, is opened 30° to the position $XYCD$, as shown in the second diagram.

a) Write down the dimensions (length, breadth and diagonal) of the lid $XYCD$.

b) Calculate XZ, the perpendicular height of X above the base of the box.

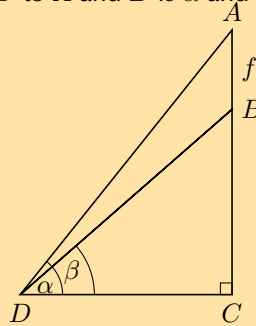
c) Calculate the ratio $\frac{\sin X\hat{Z}C}{\sin X\hat{C}Z}$.

21. AB is a vertical pole on a horizontal plane BCD. DC is a metres and the angle of elevation from D to A is θ. $A\hat{C}D = \alpha$ and $A\hat{D}C = \beta$.

a) Name the two right angles in the diagram.

b) Show that $AB = \frac{a \sin\alpha \sin\theta}{\sin(\alpha+\beta)}$.

c) If it is given that $AD = AC$, show that the height of the pole is given by $AB = \frac{a \sin\theta}{2\cos\alpha}$.

d) Calculate the height of the pole if $a = 13$ m, $\theta = 33°, \alpha = \beta = 65°$.

22. AB is a flagpole on top of a government building BC. $AB = f$ units and D is a point on the ground in the same horizontal plane as the base of the building, C. The angle of elevation from D to A and B is α and β, respectively.

a) Show that $f = \frac{BC \sin(\alpha-\beta)}{\sin\beta\cos\alpha}$

b) Calculate the height of the flagpole (to the nearest metre) if the building is 7 m, $\alpha = 63°$ and $\beta = 57°$.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 28PV	1b. 28PW	1c. 28PX	1d. 28PY	1e. 28PZ	2. 28Q2
3a. 28Q3	3b. 28Q4	4. 28Q5	5. 28Q6	6. 28Q7	7a. 28Q8
7b. 28Q9	7c. 28QB	7d. 28QC	8. 28QD	9a. 28QF	9b. 28QG
9c. 28QH	9d. 28QJ	9e. 28QK	10. 28QM	11. 28QN	12. 28QP
13. 28QQ	14. 28QR	15. 28QS	16. 28QT	17. 28QV	18. 28QW
19. 28QX	20. 28QY	21. 28QZ	22. 28R2		

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CHAPTER

Polynomials

5.1 Revision EMCGQ

Identifying polynomials EMCGR

Terminology:	
Polynomial	An expression that involves one or more variables having different powers and coefficients. $a_n x^n + \ldots + a_2 x^2 + a_1 x + a_0,$ where $n \in \mathbb{N}_0$
Monomial	A polynomial with one term. For example, $7a^2b$ or $15xyz^2$.
Binomial	A polynomial that has two terms. For example, $2x + 5z$ or $26 - g^2k$.
Trinomial	A polynomial that has three terms. For example, $a - b + c$ or $4x^2 + 17xy - y^3$.
Degree/Order	The degree, also called the order, of a univariate polynomial is the value of the highest exponent in the polynomial. For example, $7p - 12p^2 + 3p^5 + 8$ has a degree of 5.

It is important to notice that the definition of a polynomial states that all exponents of the variables must be elements of the set of natural numbers. If an expression contains terms with exponents that are not natural numbers, then it is not a polynomial.

The following examples are not polynomials:

$$\frac{3}{y} - 4y^2 + 1$$
$$5\sqrt{k} + k - 2k^2$$
$$x^{-2} + 3 + 7x^2$$
$$t^2 - 4t + 6t^{\frac{1}{3}}$$

Investigation: More on polynomials

Discuss whether the following statements are true or false:

1. The expression $3y^2 + 2y - 4$ is a trinomial of degree 2.
2. $25z^5 - 36\sqrt{z}$ is a binomial of order 5.
3. 25 is a constant polynomial of degree 0.
4. $3x^2 - 2x - 5$ is a quadratic polynomial.
5. The expression $23b^{-2}$ is a monomial because it only has one term.
6. 0 is a constant polynomial of undefined degree.
7. A cubic polynomial has three terms and all the exponents are natural numbers.
8.

$$\text{Given the expression } : \frac{1}{t} - 3t^2 + 1$$
$$\text{If we multiply by } t : 1 - 3t^3 + t,$$
$$\text{we get a trinomial of degree } 3$$

Exercise 5 – 1: Identifying polynomials

1. Given $f(x) = 2x^3 + 3x^2 - 1$, determine whether the following statements are true or false. If false, provide the correct statement.
   a) $f(x)$ is a trinomial.
   b) The coefficient of the x is zero.
   c) $f\left(\frac{1}{2}\right) = \frac{1}{12}$.
   d) $f(x)$ is of degree 3.
   e) The constant term is 1.
   f) $f(x)$ will have 3 real roots.
2. Given $g(x) = 2x^3 - 9x^2 + 7x + 6$, determine the following:
   a) the number of terms in $g(x)$.
   b) the degree of $g(x)$.
   c) the coefficient of the x^2 term.
   d) the constant term.
3. Determine which of the following expressions are polynomials and which are not. For those that are not polynomials, give reasons.
   a) $y^3 + \sqrt{5}$
   b) $-x^2 - x - 1$
   c) $4\sqrt{k} - 9$
   d) $\frac{2}{p} + p + 3$
   e) $x(x-1)(x-2) - 2$
   f) $(\sqrt{m} - 1)(\sqrt{m} + 1)$
   g) $t^0 - 1$
   h) $16y^7$
   i) $-\frac{x^3}{2} + 5x^2 + \frac{x}{3} - 11$
   j) $4b^0 + 3b^{-1} + 5b^2 - b^3$
4. Peter's mathematics homework is shown below. Find and correct his mistakes.
   Homework:
   Given $p(x) = x + \dfrac{4}{x} - 5$, answer the following questions:
   a) Simplify the expression.
   b) Is $p(x)$ a polynomial?
   c) What is the coefficient of the x term?

   Peter's answers:

   a)
   $$\begin{aligned} p(x) &= x + \tfrac{4}{x} - 5 && \text{(restrictions: } x \neq 0) \\ &= x^2 + 4 - 5x && \text{(multiply through by } x) \\ &= x^2 - 5x + 4 && \text{(write in descending order)} \\ &= (x-1)(x+4) && \text{(factorise, quadratic has two roots)} \end{aligned}$$

   b) Yes, because it can be simplified to have exponents that are all natural numbers. It is a quadratic binomial because the highest exponent is two and there are only two terms; $(x - 1)$ and $(x + 4)$.
   c) Before I simplified, the coefficient of the x term was nothing and after I simplified it became 5.
5. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 28R3 2. 28R4 3a. 28R5 3b. 28R6 3c. 28R7 3d. 28R8
3e. 28R9 3f. 28RB 3g. 28RC 3h. 28RD 3i. 28RF 3j. 28RG
4. 28RH

In earlier grades we learnt the following useful techniques and methods for factorising an expression:

- taking out a common factor
- factorising the difference of two squares
- grouping in pairs
- factorising the sum and difference of two cubes

We also looked at the different methods for factorising quadratic expressions:

- factorising by inspection
- completing the square
- using the quadratic formula
- making a suitable substitution

It is important to revise these methods; we use the quadratic formula for factorising cubic polynomials and we also use completing the square to find the equation of a circle in Chapter 7.

Terminology:	
Variable	A symbol used to represent an unknown numerical value. For example, $a, b, x, y, \alpha, \theta$.
Coefficient	The number or parameter that is multiplied by the variable of an expression.
Expression	A term or group of terms consisting of numbers, variables and the basic operators $(+, -, \times, \div)$.
Univariate expression	An expression containing only one variable.
Equation	A mathematical statement that asserts that two expressions are equal.
Identity	A mathematical relationship that equates one expression with another.
Solution	A value or set of values that satisfies the original problem statement.
Root/Zero	A root, also referred to as the "zero", of an equation is the value of x such that $f(x) = 0$ is satisfied.

Worked example 1: Solving quadratic equations using factorisation

QUESTION

Solve for x:

$$\frac{3x}{x+2} + 1 = \frac{4}{x+1}$$

SOLUTION

Step 1: Determine the restrictions

The restrictions are the values for x that would result in the denominator being equal to 0, which would make the fraction undefined. Therefore $x \neq -2$ and $x \neq -1$.

Step 2: Determine the lowest common denominator

The lowest common denominator is $(x+2)(x+1)$.

Step 3: Multiply each term in the equation by the lowest common denominator and simplify

$$\begin{aligned}\frac{3x(x+2)(x+1)}{x+2} + (x+2)(x+1) &= \frac{4(x+2)(x+1)}{x+1}\\ 3x(x+1)+(x+2)(x+1) &= 4(x+2)\\ 3x^2+3x+x^2+3x+2 &= 4x+8\\ 4x^2+2x-6 &= 0\\ 2x^2+x-3 &= 0\end{aligned}$$

Step 4: Factorise and solve the equation

$$\begin{aligned}(2x+3)(x-1) &= 0\\ 2x+3 = 0 &\text{ or } x-1=0\\ x = -\frac{3}{2} &\text{ or } x=1\end{aligned}$$

Step 5: Check the solution by substituting both answers back into the original equation

Step 6: Write the final answer

Therefore $x = -1\frac{1}{2}$ or $x = 1$.

Worked example 2: Using the quadratic formula

QUESTION

Find the roots of the function $f(x) = 3x^2 + 4x - 8$.

SOLUTION

Step 1: Finding the roots

To determine the roots of $f(x)$, we let $3x^2 + 4x - 8 = 0$.

Step 2: Check whether the expression can be factorised

The expression cannot be factorised, so the general quadratic formula must be used.

Step 3: Identify the coefficients to substitute into the formula

$$a = 3; \qquad b = 4; \qquad c = -8$$

Step 4: Apply the quadratic formula

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(4) \pm \sqrt{(4)^2 - 4(3)(-8)}}{2(3)} \\ &= \frac{-4 \pm \sqrt{16 + 96}}{6} \\ &= \frac{-4 \pm \sqrt{112}}{6} \\ &= \frac{-4 \pm \sqrt{16 \times 7}}{6} \\ &= \frac{-4 \pm 4\sqrt{7}}{6} \\ &= \frac{-2 \pm 2\sqrt{7}}{3} \end{aligned}$$

Step 5: Write the final answer

Therefore $x = \frac{-2 + 2\sqrt{7}}{3}$ or $x = \frac{-2 - 2\sqrt{7}}{3}$

Worked example 3: Solving quadratic equations by completing the square

QUESTION

Solve by completing the square: $y^2 - 10y - 11 = 0$

SOLUTION

Step 1: The equation is already in the form $ax^2 + bx + c = 0$

Step 2: Make sure the coefficient of the y^2 **term is equal to** 1

$$y^2 - 10y - 11 = 0$$

Step 3: Take half the coefficient of the y **term and square it, then add and subtract it from the equation**

The coefficient of the y term is -10. Half of the coefficient of the y term is -5 and the square of it is 25. Therefore $y^2 - 10y + 25 - 25 - 11 = 0$.

Step 4: Write the trinomial as a perfect square

$$\begin{aligned} (y^2 - 10y + 25) - 25 - 11 &= 0 \\ (y - 5)^2 - 36 &= 0 \end{aligned}$$

Step 5: Method 1: Take square roots on both sides of the equation

$$(y-5)^2 = 36$$
$$y-5 = \pm\sqrt{36}$$

Important: When taking a square root always remember that there is a positive and negative answer, since $(6)^2 = 36$ and $(-6)^2 = 36$.

$$y-5 = \pm 6$$

Step 6: Solve for y

$$\text{If } y-5 = 6$$
$$y = 11$$
$$\text{Or if } y-5 = -6$$
$$y = -1$$

Therefore $y = 11$ or $y = -1$.

Step 7: Method 2: Factorise the expression as a difference of two squares

$$(y-5)^2 - (6)^2 = 0$$
$$[(y-5)+6]\,[(y-5)-6] = 0$$

Step 8: Simplify and solve for y

$$(y+1)(y-11) = 0$$
$$\therefore y = -1 \text{ or } y = 11$$

Step 9: Write the final answer

$$y = -1 \text{ or } y = 11$$

Notice that both methods produce the same answer. These roots are rational because 36 is a perfect square.

Exercise 5 – 2: Quadratic polynomials

1. Solve the following quadratic equations by factorisation. Answers may be left in surd form, where applicable.

   a) $7p^2 + 14p = 0$
   b) $k^2 + 5k - 36 = 0$
   c) $400 = 16h^2$
   d) $(x - 1)(x + 10) + 24 = 0$
   e) $y^2 - 5ky + 4k^2 = 0$

2. Solve the following equations by completing the square:

   a) $p^2 + 10p - 2 = 0$
   b) $2(6y + y^2) = -4$
   c) $x^2 + 5x + 9 = 0$
   d) $f^2 + 30 = 2(10 - 8f)$
   e) $3x^2 + 6x - 2 = 0$

3. Solve the following using the quadratic formula.

   a) $3m^2 + m - 4 = 0$
   b) $2t^2 + 6t + 5 = 0$
   c) $y^2 - 4y + 2 = 0$
   d) $3f - 2 = -2f^2$

4. Factorise the following:

   a) $27p^3 - 1$
   b) $16 + \frac{2}{x^3}$

5. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 28RJ 1b. 28RK 1c. 28RM 1d. 28RN 1e. 28RP 2a. 28RQ
2b. 28RR 2c. 28RS 2d. 28RT 2e. 28RV 3a. 28RW 3b. 28RX
3c. 28RY 3d. 28RZ 4a. 28S2 4b. 28S3

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5.2 Cubic polynomials EMCGT

Investigation: Simple division

Consider the following and answer the questions below:

1. 6 students are at a product promotion and there are 15 free gifts to be given away. Each student must receive the same number of gifts.

   a) Determine how many gifts each student will get?
   b) How many gifts will be left over?
   c) Use the following variables to express the above situation as a mathematical equation:

      - $a =$ total number of gifts
      - $b =$ total number of students
      - $q =$ number of gifts for each student
      - $r =$ number of gifts left over

2. A group of students go to dinner together at a restaurant and the total bill is R 510. Each student contributes R 120 towards the bill. They count the money and find that they are still R 30 short.

   a) Assign variables to the known and unknown values.

   b) Write a mathematical equation to describe the situation.

   c) Use this equation to determine how many students went to dinner.

We know that 11 divided by 2 gives an answer of 5 with a remainder 1.

$$\frac{11}{2} = 5 \text{ remainder } 1$$
$$\frac{11}{2} = 5 + \frac{1}{2}$$

This means that:

We can write a general expression for the rule of division; if an integer a is divided by an integer b, then the answer is q with a remainder r.

$$\frac{a}{b} = q + \frac{r}{b}$$
$$a = b \times q + r$$

where $b \neq 0$ and $0 \leq r < b$.

This rule can be extended to include the division of polynomials; if a polynomial $a(x)$ is divided by a polynomial $b(x)$, then the answer is $Q(x)$ with a remainder $R(x)$.

$$a(x) = b(x) \times Q(x) + R(x)$$

where $b(x) \neq 0$.

In words: the dividend is equal to the divisor multiplied by the quotient, plus the remainder.

A cubic polynomial is an expression with the highest power equal to 3; we say that the degree of the polynomial is 3.

The general form of a cubic polynomial is

$$y = ax^3 + bx^2 + cx + d$$

where $a \neq 0$.

In Grade 10 we learnt how to factorise the sum and difference of two cubes by first finding a factor (the first bracket) and then using inspection (the second bracket). For example,

$$p^3 + 8 = (p+2)(p^2 - 2p + 4)$$
$$k^3 - 1 = (k-1)(k^2 + k + 1)$$

In this section we focus on factorising cubic polynomials with one variable (univariate) where b and c are not zero.

We use the following methods for factorising cubic polynomials:

- long division
- synthetic division
- inspection

Worked example 4: Long division

QUESTION

Use the long division method to determine the quotient $Q(x)$ and the remainder $R(x)$ if $a(x) = 2x^3 - x^2 - 6x + 16$ is divided by $b(x) = x - 1$. Write your answer in the form $a(x) = b(x) \,.\, Q(x) + R(x)$.

SOLUTION

Step 1: Write down the known and unknown expressions

$$\begin{aligned} a(x) &= b(x) \,.\, Q(x) + R(x) \\ 2x^3 - x^2 - 6x + 16 &= (x-1) \,.\, Q(x) + R(x) \end{aligned}$$

Step 2: Use long division method to determine $Q(x)$ and $R(x)$

Make sure that $a(x)$ and $b(x)$ are written in descending order of the exponents. If a term of a certain degree is missing from $a(x)$, then write the term with a coefficient of 0.

$$\begin{array}{r} 2x^2 + x - 5 \\ x - 1 | \overline{2x^3 - x^2 - 6x + 16} \\ -\left(\underline{2x^3 - 2x^2}\right) \\ 0 + x^2 - 6x \\ -\left(\underline{x^2 - x}\right) \\ 0 - 5x + 16 \\ -\left(\underline{-5x + 5}\right) \\ 0 + 11 \end{array}$$

Step 3: Write the final answer

$$\begin{aligned} Q(x) &= 2x^2 + x - 5 \\ R(x) &= 11 \\ \text{and } a(x) &= b(x) \,.\, Q(x) + R(x) \\ \therefore a(x) &= (x-1)(2x^2 + x - 5) + 11 \end{aligned}$$

Synthetic division is a simpler and more efficient method for dividing polynomials. It allows us to determine the quotient and the remainder by considering the coefficients of the terms in each of the polynomials without needing to rewrite the variable and exponent for each term. If a term of a certain degree is missing from $a(x)$, then write the term with a coefficient of 0. For example, $a(x) = 5x^3 + 6x - 1$ should be written as $a(x) = 5x^3 + 0x^2 + 6x - 1$.

Notice that for synthetic division:

- the coefficients of the dividend ($a(x)$) are written below the horizontal line.
- the coefficients of the quotient ($Q(x)$) are written above the horizontal line.
- we add coefficients instead of subtracting as is the case with long division
- we use the opposite sign of the divisor ($b(x)$); the divisor is $(x - 1)$ and we use $+1$.
- the coefficient of the x term in the divisor is 1, so $q_2 = a_3$.

See video: 28S4 at www.everythingmaths.co.za

Worked example 5: Synthetic division

QUESTION

Use the synthetic division method to determine the quotient $Q(x)$ and the remainder $R(x)$ if $a(x) = 2x^3 - x^2 - 6x + 16$ is divided by $b(x) = x - 1$. Write your answer in the form $a(x) = b(x)\ .\ Q(x) + R(x)$.

SOLUTION

Step 1: Write down the known and unknown expressions

$$\begin{aligned} a(x) &= b(x)\ .\ Q(x) + R(x) \\ 2x^3 - x^2 - 6x + 16 &= (x - 1)\ .\ Q(x) + R(x) \end{aligned}$$

Step 2: Use synthetic division to determine $Q(x)$ and $R(x)$

$$\begin{array}{r|rrrr} & 2 & +1 & -5 & 11 \\ \hline 1 & 2 & -1 & -6 & 16 \end{array}$$

$$\begin{aligned} q_2 &= 2 \\ q_1 &= -1 + (2)(1) = 1 \\ q_0 &= -6 + (1)(1) = -5 \\ R &= 16 + (-5)(1) = 11 \end{aligned}$$

Step 3: Write the final answer

The quotient will be one degree lower than the dividend if we divide by a linear expression, therefore we have:

$$\begin{aligned} Q(x) &= 2x^2 + x - 5 \\ R(x) &= 11 \\ \text{and } a(x) &= b(x)\ .\ Q(x) + R(x) \\ \therefore a(x) &= (x - 1)(2x^2 + x - 5) + 11 \end{aligned}$$

General method: given the dividend $a(x) = a_3x^3 + a_2x^2 + a_1x^1 + a_0x^0$ and the divisor $(cx - d)$, we determine the quotient $Q(x) = q_2x^2 + q_1x^1 + q_0x^0$ and the remainder $R(x)$ using:

$$\begin{array}{c|cccc} & q_2 & q_1 & q_0 & R \\ \hline \frac{d}{c} & a_3 & a_2 & a_1 & a_0 \end{array}$$

We determine the coefficients of the quotient by calculating:

$$\begin{aligned} q_2 &= a_3 + \left(q_3 \times \frac{d}{c}\right) \\ &= a_3 \quad \text{(since } q_3 = 0) \\ q_1 &= a_2 + \left(q_2 \times \frac{d}{c}\right) \\ q_0 &= a_1 + \left(q_1 \times \frac{d}{c}\right) \\ R &= a_0 + \left(q_0 \times \frac{d}{c}\right) \end{aligned}$$

Important note: $a(x)$ is a function and $a_3, a_2, a_1,$ and a_0 are coefficients.

See video: 28S5 at www.everythingmaths.co.za

Worked example 6: Synthetic division

QUESTION

Use the synthetic division method to determine the quotient $Q(x)$ and the remainder $R(x)$ if $a(x) = 6x^3 + x^2 - 4x + 5$ is divided by $b(x) = 2x - 1$.

SOLUTION

Step 1: Write down the known and unknown expressions

$$\begin{aligned} a(x) &= b(x)\ .\ Q(x) + R(x) \\ 6x^3 + x^2 - 4x + 5 &= (2x - 1)\ .\ Q(x) + R(x) \end{aligned}$$

Step 2: Use synthetic division to determine $Q(x)$ and $R(x)$

Make the leading coefficient of the divisor equal to 1:

$$b(x) = (2x - 1) = 2\left(x - \frac{1}{2}\right)$$

$$\begin{array}{c|cccc} & 6 & 4 & -2 & 4 \\ \hline \frac{1}{2} & 6 & 1 & -4 & 5 \end{array}$$

$$q_2 = 6$$

$$q_1 = 1 + (6)\left(\frac{1}{2}\right) = 4$$

$$q_0 = -4 + (4)\left(\frac{1}{2}\right) = -2$$

$$R = 5 + (-2)\left(\frac{1}{2}\right) = 4$$

Step 3: Write the final answer

$$\begin{aligned}
Q(x) &= 6x^2 + 4x - 2 \\
&= 2\left(3x^2 + 2x - 1\right) \\
R &= 4 \\
\text{and } a(x) &= \frac{1}{2}b(x)\ .\ Q(x) + R(x) \\
\therefore a(x) &= \frac{1}{2}\ .\ 2\left(x - \frac{1}{2}\right)(6x^2 + 4x - 2) + 4 \\
&= \frac{1}{2}\ .\ 2\left(x - \frac{1}{2}\right)(2)(3x^2 + 2x - 1) + 4 \\
&= (2x - 1)(3x^2 + 2x - 1) + 4
\end{aligned}$$

See video: 28S6 at www.everythingmaths.co.za

Exercise 5 – 3: Cubic polynomials

1. Factorise the following:

   a) $p^3 - 1$
   b) $t^3 + 27$
   c) $64 - m^3$
   d) $k - 125k^4$
   e) $8a^6 - b^9$
   f) $8 - (p + q)^3$

2. For each of the following:

   - Use long division to determine the quotient $Q(x)$ and the remainder $R(x)$.
   - Write $a(x)$ in the form $a(x) = b(x)\ .\ Q(x) + R(x)$.
   - Check your answer by expanding the brackets to get back to the original cubic polynomial.

   a) $a(x) = x^3 + 2x^2 + 3x + 7$ is divided by $(x + 1)$
   b) $a(x) = 1 + 4x^2 - 5x - x^3$ and $b(x) = x + 2$
   c) $a(x) = 2x^3 + 3x^2 + x - 6$ and $b(x) = x - 1$

d) $a(x) = x^3 + 2x^2 + 5$ and $b(x) = x - 1$

e) $(x - 1)$ is divided into $a(x) = x^4 + 2x^3 - 3x^2 + 5x + 4$

f) $\frac{a(x)}{b(x)} = \frac{5x^4+3x^3+6x^2+x+2}{x^2-2}$

g) $a(x) = 3x^3 - x^2 + 2x + 1$ is divided by $(3x - 1)$

h) $a(x) = 2x^5 + x^3 + 3x^2 - 4$ and $b(x) = x + 2$

3. Use synthetic division to determine the quotient $Q(x)$ and the remainder $R(x)$ when $f(x)$ is divided by $g(x)$:

a) $f(x) = x^2 + 5x + 1$, $g(x) = x + 2$

b) $f(x) = x^2 - 5x - 7$, $g(x) = x - 1$

c) $f(x) = 2x^3 + 5x - 4$, $g(x) = x - 1$

d) $f(x) = 19 + x^2 + 8x$, $g(x) = x + 3$

e) $f(x) = x^3 + 2x^2 + x - 10$, $g(x) = x - 1$

f) $f(x) = 2x^3 + 7x^2 + 2x - 3$, $g(x) = x + 3$

g) $f(x) = 4x^3 + 4x^2 - x - 2$, $g(x) = 2x - 1$

h) $f(x) = 5x + 22 + 2x^3 + x^2$, $g(x) = 2x + 3$

4. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 28S7	1b. 28S8	1c. 28S9	1d. 28SB	1e. 28SC	1f. 28SD
2a. 28SF	2b. 28SG	2c. 28SH	2d. 28SJ	2e. 28SK	2f. 28SM
2g. 28SN	2h. 28SP	3a. 28SQ	3b. 28SR	3c. 28SS	3d. 28ST
3e. 28SV	3f. 28SW	3g. 28SX	3h. 28SY		

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5.3 Remainder theorem EMCGV

Investigation: Remainder theorem

Given the following functions:

- $f(x) = x^3 + 3x^2 + 4x + 12$
- $k(x) = x - 1$
- $g(x) = 4x^3 - 2x^2 + x - 7$
- $h(x) = x + 2$

1. Determine $\frac{f(x)}{k(x)}$ and $\frac{g(x)}{h(x)}$.
2. Write your answers in the general form: $a(x) = b(x).Q(x) + R(x)$.
3. Determine $f(1)$ and $g(-2)$.
4. What do you notice?
5. Consider the degree of the quotient and the remainder - is there a rule?
6. What conclusions can you draw?
7. Write a mathematical equation to describe your conclusions.
8. Complete the following sentence: a cubic function divided by a linear polynomial gives a quotient with a degree of ...... and a remainder with a degree of ......, which is called a constant.

The Remainder theorem

A polynomial $p(x)$ divided by $cx - d$ gives a remainder of $p\left(\frac{d}{c}\right)$.

In words: the value of the remainder R is obtained by substituting $x = \frac{d}{c}$ into the polynomial $p(x)$.

$$R = p\left(\frac{d}{c}\right)$$

NOTE: PROOF NOT FOR EXAMS

Let the quotient be $Q(x)$ and let the remainder be R. Therefore we can write:

$$\begin{aligned}
p(x) &= (cx - d) \,.\, Q(x) + R \\
\therefore p\left(\frac{d}{c}\right) &= \left[c\left(\frac{d}{c}\right) - d\right] \,.\, Q\left(\frac{d}{c}\right) + R \\
&= (d - d) \,.\, Q\left(\frac{d}{c}\right) + R \\
&= 0 \,.\, Q\left(\frac{d}{c}\right) + R \\
&= R \\
\therefore p\left(\frac{d}{c}\right) &= R
\end{aligned}$$

Worked example 7: Finding the remainder

QUESTION

Use the remainder theorem to determine the remainder when $p(x) = 3x^3 + 5x^2 - x + 1$ is divided by the following linear polynomials:

1. $x + 2$
2. $2x - 1$
3. $x + m$

SOLUTION

Step 1: Determine the remainder for each linear divisor

The remainder theorem states that any polynomial $p(x)$ that is divided by $cx - d$ gives a remainder of $p\left(\frac{d}{c}\right)$:

1.

$$\begin{aligned} p(x) &= 3x^3 + 5x^2 - x + 1 \\ p(-2) &= 3(-2)^3 + 5(-2)^2 - (-2) + 1 \\ &= 3(-8) + 5(4) + 2 + 1 \\ &= -24 + 20 + 3 \\ \therefore R &= -1 \end{aligned}$$

2.

$$\begin{aligned} p(x) &= 3x^3 + 5x^2 - x + 1 \\ p\left(\frac{1}{2}\right) &= 3\left(\frac{1}{2}\right)^3 + 5\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right) + 1 \\ &= 3\left(\frac{1}{8}\right) + 5\left(\frac{1}{4}\right) - \left(\frac{1}{2}\right) + 1 \\ &= \frac{3}{8} + \frac{5}{4} + \frac{1}{2} \\ &= \frac{3}{8} + \frac{10}{8} + \frac{4}{8} \\ \therefore R &= \frac{17}{8} \end{aligned}$$

3.

$$\begin{aligned} p(x) &= 3x^3 + 5x^2 - x + 1 \\ p(m) &= 3(-m)^3 + 5(-m)^2 - (-m) + 1 \\ \therefore R &= -3m^3 + 5m^2 + m + 1 \end{aligned}$$

Worked example 8: Using the remainder to solve for an unknown variable

QUESTION

Given that $f(x) = 2x^3 + x^2 + kx + 5$ divided by $2x - 3$ gives a remainder of $9\frac{1}{2}$, use the remainder theorem to determine the value of k.

SOLUTION

Step 1: Use the remainder theorem to determine the unknown variable k

From the remainder theorem we know that $f\left(\frac{3}{2}\right) = 9\frac{1}{2}$ and we can therefore solve for k:

$$\begin{aligned} f(x) &= 2x^3 + x^2 + kx + 5 \\ f\left(\frac{3}{2}\right) &= 2\left(\frac{3}{2}\right)^3 + \left(\frac{3}{2}\right)^2 + k\left(\frac{3}{2}\right) + 5 \\ 9\frac{1}{2} &= 2\left(\frac{27}{8}\right) + \left(\frac{9}{4}\right) + k\left(\frac{3}{2}\right) + 5 \\ 9\frac{1}{2} &= \frac{27}{4} + \frac{9}{4} + \frac{3k}{2} + 5 \\ 9\frac{1}{2} &= \frac{36}{4} + \frac{3k}{2} + 5 \\ \therefore 9\frac{1}{2} - 9 - 5 &= \frac{3k}{2} \\ -4\frac{1}{2} &= \frac{3k}{2} \\ -\frac{9}{2} \times \frac{2}{3} &= k \\ \therefore -3 &= k \end{aligned}$$

Step 2: Write the final answer

Therefore $k = -3$ and $f(x) = 2x^3 + x^2 - 3x + 5$.

Exercise 5 – 4: Remainder theorem

1. Use the remainder theorem to determine the remainder R when $g(x)$ is divided by $h(x)$:

   a) $g(x) = x^3 + 4x^2 + 11x - 5$
   $h(x) = x - 1$

   b) $g(x) = 2x^3 - 5x^2 + 8$
   $h(x) = 2x - 1$

   c) $g(x) = 4x^3 + 5x^2 + 6x - 1$
   $h(x) = x + 2$

   d) $g(x) = -5x^3 - x^2 - 10x + 9$
   $h(x) = 5x + 1$

   e) $g(x) = x^4 + 5x^2 + 2x - 8$
   $h(x) = x + 1$

   f) $g(x) = 3x^5 - 8x^4 + x^2 + 2$
   $h(x) = 2 - x$

   g) $g(x) = 2x^{100} - x - 1$
   $h(x) = x + 1$

2. Determine the value of t if $x^3 + tx^2 + 8x + 21$ divided by $x + 1$ gives a remainder of 16.
3. Calculate the value of m if $2x^3 - 7x^2 + mx - 26$ is divided by $x - 2$ and gives a remainder of -24.
4. If $x^5 - 2x^3 - kx - 1$ is divided by $x - 1$ and the remainder is $-\frac{1}{2}$, find the value of k .
5. Determine the value of p if $18x^3 + px^2 - 8x + 9$ is divided by $2x - 1$ and gives a remainder of 6.
6. If $x^3 + x^2 - x + b$ is divided by $x - 2$ and the remainder is $2\frac{1}{2}$, calculate the value of b.
7. Calculate the value of h if $3x^5 + hx^4 + 10x^2 - 21x + 12$ is divided by $x - 2$ and gives a remainder of 10.
8. If $x^3 + 8x^2 + mx - 5$ is divided by $x + 1$ and the remainder is n, express m in terms of n.
9. When the polynomial $2x^3 + px^2 + qx + 1$ is divided by $x + 1$ or $x - 4$, the remainder is 5. Determine the values of p and q.
10. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 28SZ 1b. 28T2 1c. 28T3 1d. 28T4 1e. 28T5 1f. 28T6
1g. 28T7 2. 28T8 3. 28T9 4. 28TB 5. 28TC 6. 28TD
7. 28TF 8. 28TG 9. 28TH

5.4 Factor theorem EMCGW

If an integer a is divided by an integer b, and the answer is q with the remainder $r = 0$, then we know that b is a factor of a.

$$\begin{aligned} a &= b \times q + r \\ \text{and } r &= 0 \\ \text{then we know that } a &= b \times q \\ \text{and also that } \frac{a}{b} &= q \end{aligned}$$

This is also true of polynomials; if a polynomial $a(x)$ is divided by a polynomial $b(x)$, and the answer is $Q(x)$ with the remainder $R(x) = 0$, then we know that $b(x)$ is a factor of $a(x)$.

$$\begin{aligned} a(x) &= b(x) \,.\, Q(x) + R(x) \\ \text{and } R(x) &= 0 \\ \text{then we know that } a(x) &= b(x) \,.\, Q(x) \\ \text{and also that } \frac{a(x)}{b(x)} &= Q(x) \end{aligned}$$

The factor theorem describes the relationship between the root of a polynomial and a factor of the polynomial.

The Factor theorem
If the polynomial $p(x)$ is divided by $cx - d$ and the remainder, given by $p\left(\frac{d}{c}\right)$, is equal to zero, then $cx - d$ is a factor of $p(x)$.

Converse: if $(cx - d)$ is a factor of $p(x)$, then $p\left(\frac{d}{c}\right) = 0$.

Worked example 9: Factor theorem

QUESTION

Using the factor theorem, show that $y + 4$ is a factor of $g(y) = 5y^4 + 16y^3 - 15y^2 + 8y + 16$.

SOLUTION

Step 1: Determine how to approach the problem
For $y + 4$ to be a factor, $g(-4)$ must be equal to 0.

Step 2: Calculate $g(-4)$

$$\begin{aligned} g(y) &= 5y^4 + 16y^3 - 15y^2 + 8y + 16 \\ \therefore g(-4) &= 5(-4)^4 + 16(-4)^3 - 15(-4)^2 + 8(-4) + 16 \\ &= 5(256) + 16(-64) - 15(16) + 8(-4) + 16 \\ &= 1280 - 1024 - 240 - 32 + 16 \\ &= 0 \end{aligned}$$

Step 3: Conclusion
Since $g(-4) = 0$, $y + 4$ is a factor of $g(y)$.

In general, to factorise a cubic polynomial we need to do the following:

- Find one factor by trial and error: consider the coefficients of the given cubic polynomial $p(x)$ and guess a possible root ($\frac{c}{d}$).
- Use the factor theorem to confirm that $\frac{c}{d}$ is a root; show that $p\left(\frac{c}{d}\right) = 0$.
- Divide $p(x)$ by the factor $(cx - d)$ to obtain a quadratic polynomial (remember to be careful with the signs).
- Apply the standard methods of factorisation to determine the two factors of the quadratic polynomial.

Worked example 10: Factor theorem

QUESTION

Use the factor theorem to determine if $y - 1$ is a factor of $f(y) = 2y^4 + 3y^2 - 5y + 7$.

SOLUTION

Step 1: Determine how to approach the problem

For $y - 1$ to be a factor, $f(1)$ must be equal to 0.

Step 2: Calculate $f(1)$

$$\begin{aligned} f(y) &= 2y^4 + 3y^2 - 5y + 7 \\ \therefore f(1) &= 2(1)^4 + 3(1)^2 - 5(1) + 7 \\ &= 2 + 3 - 5 + 7 \\ &= 7 \end{aligned}$$

Step 3: Conclusion

Since $f(1) \neq 0$, $y - 1$ is not a factor of $f(y) = 2y^4 + 3y^2 - 5y + 7$.

Worked example 11: Factorising cubic polynomials

QUESTION

Factorise completely: $f(x) = x^3 + x^2 - 9x - 9$

SOLUTION

Step 1: Find a factor by trial and error

Try $f(1) = (1)^3 + (1)^2 - 9(1) - 9 = 1 + 1 - 9 - 9 = -16$

Therefore $(x - 1)$ is not a factor.

We consider the coefficients of the given polynomial and try:

$f(-1) = (-1)^3 + (-1)^2 - 9(-1) - 9 = -1 + 1 + 9 - 9 = 0$

Therefore $(x+1)$ is a factor, because $f(-1) = 0$.

Step 2: Factorise by inspection
Now divide $f(x)$ by $(x+1)$ using inspection:

Write $x^3 + x^2 - 9x - 9 = (x+1)(\ldots)$

The first term in the second bracket must be x^2 to give x^3 and make the polynomial a cubic.

The last term in the second bracket must be -9 because $(+1)(-9) = -9$.

So we have $x^3 + x^2 - 9x - 9 = (x+1)\left(x^2 + ?x - 9\right)$

Now, we must find the coefficient of the middle term:

$(+1)\left(x^2\right)$ gives the x^2 in the original polynomial. So, the coefficient of the x-term must be 0.

$\therefore f(x) = (x+1)\left(x^2 - 9\right)$.

Step 3: Write the final answer
We can factorise the last bracket as a difference of two squares:

$$\begin{aligned} f(x) &= (x+1)\left(x^2 - 9\right) \\ &= (x+1)(x-3)(x+3) \end{aligned}$$

Worked example 12: Factorising cubic polynomials

QUESTION

Use the factor theorem to factorise $f(x) = x^3 - 2x^2 - 5x + 6$.

SOLUTION

Step 1: Find a factor by trial and error
Try $f(1) = (1)^3 - 2(1)^2 - 5(1) + 6 = 1 - 2 - 5 + 6 = 0$

Therefore $(x-1)$ is a factor.

Step 2: Factorise by inspection
$x^3 - 2x^2 - 5x + 6 = (x-1)(\ldots)$

The first term in the second bracket must be x^2 to give x^3 if we work backwards.

The last term in the second bracket must be -6 because $(-1)(-6) = +6$.

So we have $x^3 - 2x^2 - 5x + 6 = (x-1)\left(x^2 + ?x - 6\right)$

Now, we must find the coefficient of the middle term:

$(-1)\left(x^2\right)$ gives $-x^2$. So, the coefficient of the x-term in the second bracket must be -1 to give another $-x^2$ so that overall we have $-x^2 - x^2 = -2x^2$.

So $f(x) = (x-1)\left(x^2 - x - 6\right)$.

Make sure that the expression has been factorised correctly by checking that the coefficient of the x-term also works out: $(x)(-6) + (-1)(-x) = -5x$, which is correct.

Step 3: Write the final answer

We can factorise the last bracket as:

$$\begin{aligned} f(x) &= (x-1)\left(x^2 - x - 6\right) \\ &= (x-1)(x-3)(x+2) \end{aligned}$$

Exercise 5 – 5: Factorising cubic polynomials

1. Find the remainder when $4x^3 - 4x^2 + x - 5$ is divided by $x + 1$.
2. Use the factor theorem to factorise $x^3 - 3x^2 + 4$ completely.
3. $f(x) = 2x^3 + x^2 - 5x + 2$
   a) Find $f(1)$.
   b) Factorise $f(x)$ completely.
4. Use the factor theorem to determine all the factors of the following expression:
   $x^3 + x^2 - 17x + 15$
5. Complete: If $f(x)$ is a polynomial and p is a number such that $f(p) = 0$, then $(x - p)$ is...
6. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 28TJ 2. 28TK 3a. 28TM 3b. 28TN 4. 28TP 5. 28TQ

5.5 Solving cubic equations EMCGX

Now that we know how to factorise cubic polynomials, it is also easy to solve cubic equations of the form $ax^3 + bx^2 + cx + d = 0$.

Worked example 13: Solving cubic equations

QUESTION

Solve: $6x^3 - 5x^2 - 17x + 6 = 0$

SOLUTION

Step 1: Find one factor using the factor theorem

Let $f(x) = 6x^3 - 5x^2 - 17x + 6$

Try $f(1) = 6(1)^3 - 5(1)^2 - 17(1) + 6 = 6 - 5 - 17 + 6 = -10$

Therefore $(x - 1)$ is not a factor.

Try $f(2) = 6(2)^3 - 5(2)^2 - 17(2) + 6 = 48 - 20 - 34 + 6 = 0$

Therefore $(x - 2)$ is a factor.

Step 2: Factorise by inspection

$$6x^3 - 5x^2 - 17x + 6 = (x - 2)\left(6x^2 + 7x - 3\right)$$

Step 3: Factorise fully

$$6x^3 - 5x^2 - 17x + 6 = (x - 2)(2x + 3)(3x - 1)$$

Step 4: Solve the equation

$$\begin{aligned} 6x^3 - 5x^2 - 17x + 6 &= 0 \\ (x - 2)(2x + 3)(3x - 1) &= 0 \\ x = 2 \text{ or } x &= \frac{1}{3} \text{ or } x = -\frac{3}{2} \end{aligned}$$

Sometimes it is not possible to factorise a quadratic expression using inspection, in which case we use the quadratic formula to fully factorise and solve the cubic equation.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Worked example 14: Solving cubic equations

QUESTION

Solve for x: $0 = x^3 - 2x^2 - 6x + 4$

SOLUTION

Step 1: Use the factor theorem to determine a factor

Let $f(x) = x^3 - 2x^2 - 6x + 4$

Try $f(1) = (1)^3 - 2(1)^2 - 6(1) + 4 = 1 - 2 - 6 + 4 = -3$

Therefore $(x - 1)$ is not a factor.

Try $f(2) = (2)^3 - 2(2)^2 - 6(2) + 4 = 8 - 8 - 12 + 4 = -8$

Therefore $(x - 2)$ is not a factor.

$f(-2) = (-2)^3 - 2(-2)^2 - 6(-2) + 4 = -8 - 8 + 12 + 4 = 0$

Therefore $(x + 2)$ is a factor.

Step 2: Factorise by inspection

$$x^3 - 2x^2 - 6x + 4 = (x + 2)\left(x^2 - 4x + 2\right)$$

$x^2 - 4x + 2$ cannot be factorised any further and we are left with

$(x + 2)\left(x^2 - 4x + 2\right) = 0$

Step 3: Solve the equation

$$(x + 2)\left(x^2 - 4x + 2\right) = 0$$
$$(x + 2) = 0 \text{ or } \left(x^2 - 4x + 2\right) = 0$$

Step 4: Apply the quadratic formula for the second bracket

Always write down the formula first and then substitute the values of a, b and c.

$$a = 1; \qquad b = -4; \qquad c = 2$$
$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(2)}}{2(1)} \\ &= \frac{4 \pm \sqrt{8}}{2} \\ &= 2 \pm \sqrt{2} \end{aligned}$$

Step 5: Final solutions

$x = -2$ or $x = 2 \pm \sqrt{2}$

Exercise 5 – 6: Solving cubic equations

Solve the following cubic equations:

1. $x^3 + x^2 - 16x = 16$
2. $-n^3 - n^2 + 22n + 40 = 0$
3. $y(y^2 + 2y) = 19y + 20$
4. $k^3 + 9k^2 + 26k + 24 = 0$
5. $x^3 + 2x^2 - 50 = 25x$
6. $-p^3 + 19p = 30$
7. $6x^2 - x^3 = 5x + 12$

Check answers online with the exercise code below or click on 'show me the answer'.
1. 28TR 2. 28TS 3. 28TT 4. 28TV 5. 28TW 6. 28TX
7. 28TY

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5.6 Summary EMCGY

See presentation: 28TZ at www.everythingmaths.co.za

Terminology:	
Expression	A term or group of terms consisting of numbers, variables and the basic operators $(+, -, \times, \div)$.
Univariate expression	An expression containing only one variable.
Root/Zero	A root, also referred to as the "zero", of an equation is the value of x such that $f(x) = 0$ is satisfied.
Polynomial	An expression that involves one or more variables having different powers and coefficients. $a_n x^n + \ldots + a_2 x^2 + a_1 x + a_0$, where $n \in \mathbb{N}_0$
Monomial	A polynomial with one term. For example, $7a^2b$ or $15xyz^2$.
Binomial	A polynomial that has two terms. For example, $2x + 5z$ or $26 - g^2k$.
Trinomial	A polynomial that has three terms. For example, $a - b + c$ or $4x^2 + 17xy - y^3$.
Degree/Order	The degree, also called the order, of a univariate polynomial is the value of the highest exponent in the polynomial. For example, $7p - 12p^2 + 3p^5 + 8$ has a degree of 5.

- Quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- Remainder theorem: a polynomial $p(x)$ divided by $cx - d$ gives a remainder of $p\left(\frac{d}{c}\right)$.
- Factor theorem: if the polynomial $p(x)$ is divided by $cx - d$ and the remainder, $p\left(\frac{d}{c}\right)$, is equal to zero, then $cx - d$ is a factor of $p(x)$.
- Converse of the factor theorem: if $cx - d$ is a factor of $p(x)$, then $p\left(\frac{d}{c}\right) = 0$.

- Synthetic division:

$$\begin{array}{c|cccc} & q_2 & q_1 & q_0 & R \\ \hline \frac{d}{c} & a_3 & a_2 & a_1 & a_0 \end{array}$$

We determine the coefficients of the quotient by calculating:

$$\begin{aligned} q_2 &= a_3 + \left(q_3 \times \frac{d}{c}\right) \\ &= a_3 \quad \text{(since } q_3 = 0) \\ q_1 &= a_2 + \left(q_2 \times \frac{d}{c}\right) \\ q_0 &= a_1 + \left(q_1 \times \frac{d}{c}\right) \\ R &= a_0 + \left(q_0 \times \frac{d}{c}\right) \end{aligned}$$

Exercise 5 – 7: End of chapter exercises

1. Solve for x: $x^3 + x^2 - 5x + 3 = 0$
2. Solve for y: $y^3 = 3y^2 + 16y + 12$
3. Solve for m: $m(m^2 - m - 4) = -4$
4. Solve for x: $x^3 - x^2 = 3\,(3x + 2)$
5. Solve for x if $2x^3 - 3x^2 - 8x = 3$.
6. Solve for x: $16\,(x + 1) = x^2\,(x + 1)$
7. a) Show that $x - 2$ is a factor of $3x^3 - 11x^2 + 12x - 4$.
   b) Hence, by factorising completely, solve the equation:
   $3x^3 - 11x^2 + 12x - 4 = 0$
8. $2x^3 - x^2 - 2x + 2 = Q\,(x)\,.\,(2x - 1) + R$ for all values of x.
   What is the value of R?
9. a) Use the factor theorem to solve the following equation for m:
   $8m^3 + 7m^2 - 17m + 2 = 0$
   b) Hence, or otherwise, solve for x:
   $2^{3x+3} + 7.2^{2x} + 2 = 17.2^x$
10. Find the value of R if $x - 1$ is a factor of $h(x) = (x - 6)\,.\,Q(x) + R$ and $Q(x)$ divided by $x - 1$ gives a remainder of 8.
11. Determine the values of p for which the function

$$f\,(x) = 3p^3 - (3p - 7)\,x^2 + 5x - 3$$

    leaves a remainder of 9 when it is divided by $(x - p)$.
12. Calculate t and $Q(x)$ if $x^2 + tx + 3 = (x + 4)\,.\,Q(x) - 17$.
13. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 28V2 2. 28V3 3. 28V4 4. 28V5 5. 28V6 6. 28V7
7. 28V8 8. 28V9 9. 28VB 10. 28VC 11. 28VD 12. 28VF

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CHAPTER

Differential calculus

6.1 Limits EMCGZ

Calculus is one of the central branches of mathematics and was developed from algebra and geometry. It is built on the concept of limits, which will be discussed in this chapter. Calculus consists of two related ideas: differential calculus and integral calculus. We will only be dealing with differential calculus in this chapter and will explore how it can be used to solve optimisation problems and finding rates of change.

The tale of Achilles and the tortoise EMCH2

Zeno (born about 490 BC) was a philosopher of southern Italy who is famous for his paradoxes (a "paradox" is a statement that seems contradictory and yet may be true).

One of Zeno's paradoxes can be summarised as:

Achilles and a tortoise agree to a race, but the tortoise is unhappy because Achilles is very fast. So, the tortoise asks Achilles for a head start. Achilles agrees to give the tortoise a 1000 m head start. Does Achilles overtake the tortoise?

To solve this problem, we start by writing:

$$\begin{aligned} \text{Achilles:} \quad & x_A = v_A t \\ \text{Tortoise:} \quad & x_T = 1000 \text{ m} + v_T t \end{aligned}$$

where

- x_A is the distance covered by Achilles
- v_A is Achilles' speed
- t is the time taken by Achilles to overtake the tortoise
- x_T is the distance covered by the tortoise
- v_T is the tortoise's speed

Achilles will overtake the tortoise when both of them have covered the same distance. If we assume that Achilles runs at 2 m.s^{-1} and the tortoise runs at $0{,}25$ m.s^{-1}, then this means that Achilles will overtake the tortoise at a time calculated as:

$$\begin{aligned} x_A &= x_T \\ v_A t &= 1000 + v_T t \\ 2t &= 1000 + 0{,}25t \\ 2 - 0{,}25t &= 1000 \\ \frac{7}{4}t &= 1000 \\ t &= \frac{4000}{7} \\ &= 571{,}43 \text{ s} \end{aligned}$$

However, Zeno looked at it as follows: Achilles takes $t = \frac{1000 \text{ m}}{2 \text{ m.s}^{-1}} = 500$ s to travel the 1000 m head start that he gave the tortoise. However, in these 500 s, the tortoise has travelled a further $x = 500 \text{ s} \times 0{,}25 \text{ m.s}^{-1} = 125$ m.

Achilles then takes another $t = \frac{125 \text{ m}}{2 \text{ m.s}^{-1}} = 62{,}5$ s to travel the 125 m. In these 62,5 s, the tortoise travels a further $x = 62{,}5 \text{ s} \times 0{,}25 \text{ m.s}^{-1} = 15{,}625$ m.

Zeno saw that Achilles would always get closer and closer but wouldn't actually overtake the tortoise.

So what does Zeno, Achilles and the tortoise have to do with calculus? Consider our earlier studies of sequences and series:

We know that the sequence $0; \frac{1}{2}; \frac{2}{3}; \frac{3}{4}; \frac{4}{5}; \ldots$ can be defined by the expression $T_n = 1 - \frac{1}{n}$ and that the terms get closer to 1 as n gets larger.

Similarly, the sequence $1; \frac{1}{2}; \frac{1}{3}; \frac{1}{4}; \frac{1}{5}; \ldots$ can be defined by the expression $T_n = \frac{1}{n}$ and the terms get closer to 0 as n gets larger.

We have also seen that an infinite geometric series can have a finite sum.

$$S_\infty = \sum_{i=1}^{\infty} a.r^{i-1} = \frac{a}{1-r} \quad \text{for } -1 < r < 1$$

where a is the first term of the series and r is the common ratio.

We see that there are some functions where the value of the function gets close to or **approaches** a certain value as the number of terms increases.

Limits EMCH3

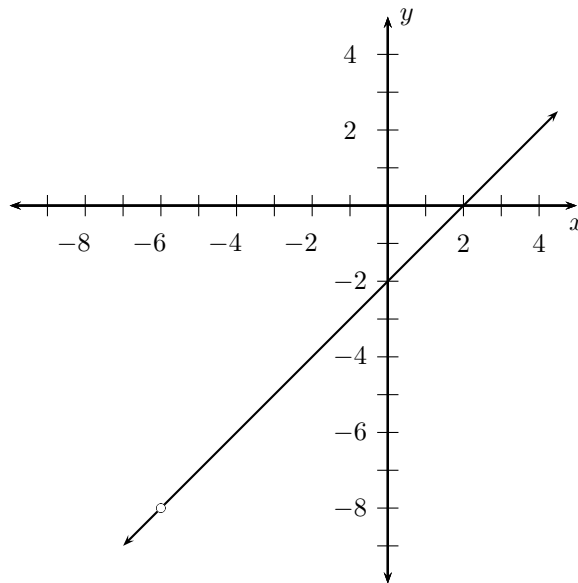
Consider the function: $y = \frac{x^2+4x-12}{x+6}$

The numerator of the function can be factorised as: $y = \frac{(x+6)(x-2)}{x+6}$.

Then we can cancel the $x + 6$ from numerator and denominator and we are left with: $y = x - 2$.

However, we are only able to cancel the $x + 6$ term if $x \neq -6$. If $x = -6$, then the denominator becomes 0 and the function is not defined. This means that the domain of the function does not include $x = -6$. But we can examine what happens to the values for y as x gets closer to -6. The list of values shows that as x gets closer to -6, y gets closer and closer to -8.

x	$y = \frac{(x+6)(x-2)}{x+6}$
-9	-11
-8	-10
-7	-9
$-6{,}5$	$-8{,}5$
$-6{,}4$	$-8{,}4$
$-6{,}3$	$-8{,}3$
$-6{,}2$	$-8{,}2$
$-6{,}1$	$-8{,}1$
$-6{,}09$	$-8{,}09$
$-6{,}08$	$-8{,}08$
$-6{,}01$	$-8{,}01$
$-5{,}9$	$-7{,}9$
$-5{,}8$	$-7{,}8$
$-5{,}7$	$-7{,}7$
$-5{,}6$	$-7{,}6$
$-5{,}5$	$-7{,}5$
-5	-7
-4	-6
-3	-5

The graph of this function is shown below. The graph is a straight line with slope 1 and y-intercept -2, but with a hole at $x = -6$. As x approaches -6 from the left, the y-value approaches -8 and as x approaches -6 from the right, the y-value approaches -8. Since the function approaches the same y-value from the left and from the right, the limit exists.

Notation EMCH4

We can now introduce new notation. For the function $y = \frac{(x+6)(x-2)}{x+6}$, we can write:

$$\lim_{x \to -6} \frac{(x+6)(x-2)}{x+6} = -8.$$

This is read: the limit of $\frac{(x+6)(x-2)}{x+6}$ as x tends to -6 (from both the left and the right) is equal to -8.

Investigation: Limits

If $f(x) = x + 1$, determine:

$f(-0{,}1)$	
$f(-0{,}05)$	
$f(-0{,}04)$	
$f(-0{,}03)$	
$f(-0{,}02)$	
$f(-0{,}01)$	
$f(0{,}00)$	
$f(0{,}01)$	
$f(0{,}02)$	
$f(0{,}03)$	
$f(0{,}04)$	
$f(0{,}05)$	
$f(0{,}1)$	

What do you notice about the value of $f(x)$ as x gets closer and closer to 0?

Worked example 1: Limit notation

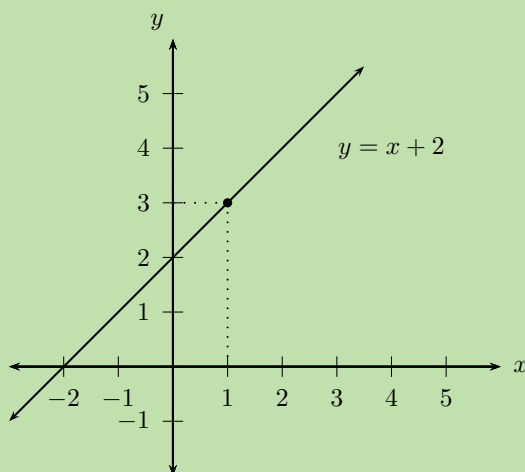
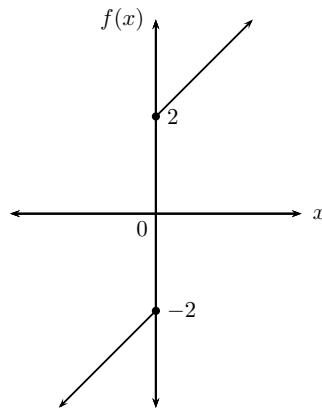
QUESTION

Write the following using limit notation: as x gets close to 1, the value of the function $y = x + 2$ approaches 3.

SOLUTION

This is written as:

$$\lim_{x \to 1} (x + 2) = 3$$

This is illustrated in the diagram below:

We can also have the situation where a function tends to a different limit depending on whether x approaches from the left or the right.

As $x \to 0$ from the left, $f(x)$ approaches -2. As $x \to 0$ from the right, $f(x)$ approaches 2.

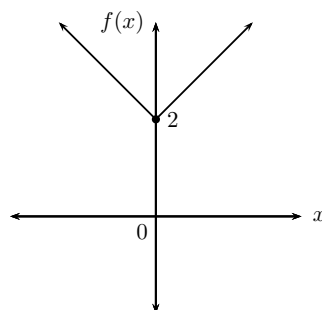
The limit for x approaching 0 from the left is:

$$\lim_{x \to 0^-} f(x) = -2$$

and for x approaching 0 from the right:

$$\lim_{x \to 0^+} f(x) = -2$$

where 0^- means x approaches zero from the left and 0^+ means x approaches zero from the right.

Therefore, since $f(x)$ does not approach the same value from both sides, we can conclude that the limit as x tends to zero does not exist.

In the diagram above, as x tends to 0 from the left, the function approaches 2 and as x tends to 0 from the right, the function approaches 2. Since the function approaches the same value from both sides, the limit as x tends to 0 exists and is equal to 2.

See video: 28VG at www.everythingmaths.co.za

Worked example 2: Limits

QUESTION

Determine:

1. $\lim_{x \to 1} 10$
2. $\lim_{x \to 2} (x + 4)$

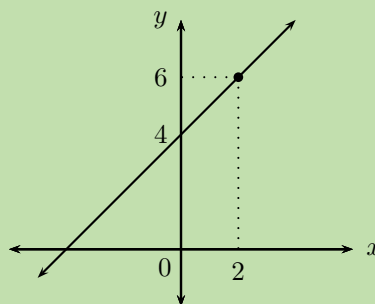
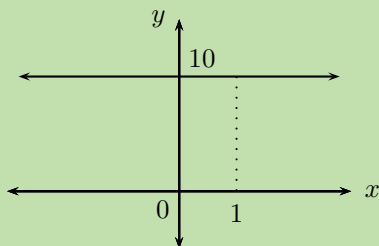
Illustrate answers graphically.

SOLUTION

Step 1: Simplify the expression and cancel all common terms

We cannot simplify further and there are no terms to cancel.

Step 2: Calculate the limit

1. $\lim_{x \to 1} 10 = 10$
2. $\lim_{x \to 2} (x + 4) = 2 + 4 = 6$

Worked example 3: Limits

QUESTION

Determine the following and illustrate the answer graphically:

$$\lim_{x \to 10} \frac{x^2 - 100}{x - 10}$$

SOLUTION

Step 1: Simplify the expression

Factorise the numerator: $\frac{x^2-100}{x-10} = \frac{(x+10)(x-10)}{x-10}$

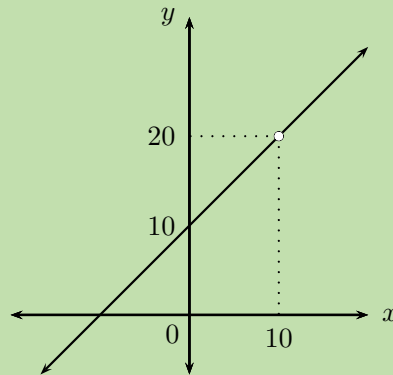
As $x \to 10$, the denominator $(x - 10) \to 0$, therefore the expression is not defined for $x = 10$ since division by zero is not permitted.

Step 2: Cancel all common terms

$$\frac{(x+10)(x-10)}{x-10} = x+10$$

Step 3: Calculate the limit

$$\begin{aligned}\lim_{x\to 10} \frac{x^2-100}{x-10} &= \lim_{x\to 10}(x+10)\\ &= 10+10\\ &= 20\end{aligned}$$

Step 4: Draw the graph

See video: 28YM at www.everythingmaths.co.za

Exercise 6 – 1: Limits

1. Determine the following limits and draw a rough sketch to illustrate:

 a) $\lim_{x\to 3} \frac{x^2-9}{x+3}$

 b) $\lim_{x\to 3} \frac{x+3}{x^2+3x}$

2. Determine the following limits (if they exist):

a) $\lim_{x \to 2} \frac{3x^2 - 4x}{3 - x}$

b) $\lim_{x \to 4} \frac{x^2 - x - 12}{x - 4}$

c) $\lim_{x \to 2} \left(3x + \frac{1}{3x}\right)$

d) $\lim_{x \to 0} \frac{1}{x}$

e) $\lim_{y \to 1} \frac{y - 1}{y + 1}$

f) $\lim_{y \to 1} \frac{y + 1}{y - 1}$

g) $\lim_{h \to 0} \frac{3h + h^2}{h}$

h) $\lim_{h \to 1} \frac{h^3 - 1}{h - 1}$

i) $\lim_{x \to 3} \frac{\sqrt{x} - \sqrt{3}}{x - 3}$

3. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 28VH 1b. 28VJ 2a. 28VK 2b. 28VM 2c. 28VN 2d. 28VP
2e. 28VQ 2f. 28VR 2g. 28VS 2h. 28VT 2i. 28VV

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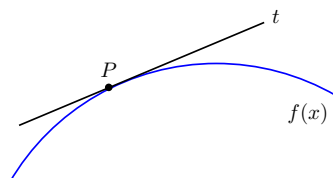
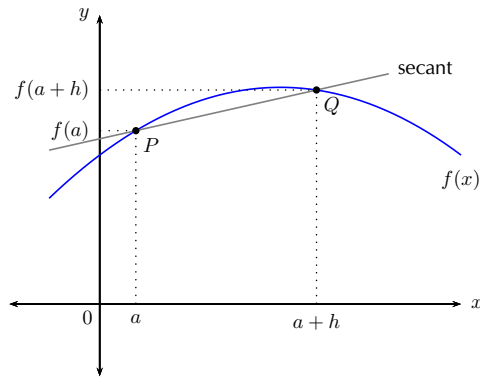
Gradient at a point — EMCH5

Average gradient

In Grade 11 we learnt that the average gradient between any two points on a curve is given by the gradient of the straight line that passes through both points. We also looked at the gradient at a single point on a curve and saw that it was the gradient of the tangent to the curve at the given point. In this section we learn how to determine the gradient of the tangent.

Let us consider finding the gradient of a tangent t to a curve with equation $y = f(x)$ at a given point P.

We know how to calculate the average gradient between two points on a curve, but we need two points. The problem now is that we only have one point, namely P. To get around the problem we first consider a secant (a straight line that intersects a curve at two or more points) to the curve that passes through point $P(x_P; y_P)$ and another point on the curve $Q(x_Q; y_Q)$, where Q is an arbitrary distance from P.

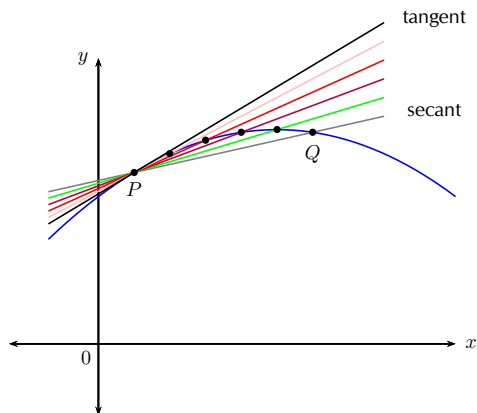
We can determine the average gradient of the curve between the two points:

$$m = \frac{y_Q - y_P}{x_Q - x_P}$$

If we let the x-coordinate of P be a, then the y-coordinate is $f(a)$. Similarly, if the x-coordinate of Q is $(a + h)$, then the y-coordinate is $f(a + h)$.

We can now calculate the average gradient as:

$$\begin{aligned}\frac{y_Q - y_P}{x_Q - x_P} &= \frac{f(a+h) - f(a)}{(a+h) - a} \\ &= \frac{f(a+h) - f(a)}{h}\end{aligned}$$

Gradient at a point

Imagine that Q moves along the curve, getting closer and closer to P. The secant line approaches the tangent line as its limiting position. This means that the average gradient of the secant approaches the gradient of the tangent to the curve at P.

We see that as point Q approaches point P, h gets closer to 0. If point Q lies on point P, then $h = 0$ and the formula for average gradient is undefined. We use our knowledge of limits to let h tend towards 0 to determine the gradient of the tangent to the curve at point P:

$$\text{Gradient at point } P = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Worked example 4: Gradient at a point

QUESTION

Given $g(x) = 3x^2$, determine the gradient of the curve at the point $x = -1$.

SOLUTION

Step 1: Write down the formula for the gradient at a point

$$\text{Gradient at a point } = \lim_{h \to 0} \frac{g(a+h) - g(a)}{h}$$

Step 2: Determine $g(a+h)$ and $g(a)$

We need to find the gradient of the curve at $x = -1$, therefore we let $a = -1$:

$$g(x) = 3x^2$$

$$\begin{aligned} g(a) &= g(-1) \\ &= 3(-1)^2 \\ &= 3 \end{aligned}$$

$$\begin{aligned} g(a+h) &= g(-1+h) \\ &= 3(-1+h)^2 \\ &= 3\left(1 - 2h + h^2\right) \\ &= 3 - 6h + 3h^2 \end{aligned}$$

Step 3: Substitute into the formula and simplify

$$\begin{aligned} \lim_{h \to 0} \frac{g(a+h) - g(a)}{h} &= \lim_{h \to 0} \frac{g(-1+h) - g(-1)}{h} \\ &= \lim_{h \to 0} \frac{\left(3 - 6h + 3h^2\right) - 3}{h} \\ &= \lim_{h \to 0} \frac{-6h + 3h^2}{h} \\ &= \lim_{h \to 0} \frac{h(-6+3h)}{h} \\ &= \lim_{h \to 0} (-6+3h) \\ &= -6 \end{aligned}$$

Notice that we only take the limit once we have removed h from the denominator.

Step 4: Write the final answer

The gradient of the curve $g(x) = 3x^2$ at $x = -1$ is -6.

Worked example 5: Gradient at a point

QUESTION

Given the function $f(x) = 2x^2 - 5x$, determine the gradient of the tangent to the curve at the point $x = 2$.

SOLUTION

Step 1: Write down the formula for the gradient at a point

$$\text{Gradient at a point } = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

Step 2: Determine $f(a+h)$ and $f(a)$

We need to find the gradient of the tangent to the curve at $x = 2$, therefore we let $a = 2$:

$$f(x) = 2x^2 - 5x$$

$$\begin{aligned} f(a) &= f(2) \\ &= 2(2)^2 - 5(2) \\ &= 8 - 10 \\ &= -2 \end{aligned}$$

$$\begin{aligned} f(a+h) &= f(2+h) \\ &= 2(2+h)^2 - 5(2+h) \\ &= 2\left(2^2 + 4h + h^2\right) - 10 - 5h \\ &= 8 + 8h + 2h^2 - 10 - 5h \\ &= -2 + 3h + 2h^2 \end{aligned}$$

Step 3: Substitute into the formula and simplify

$$\begin{aligned} \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} &= \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h\to 0} \frac{\left(-2 + 3h + 2h^2\right) - (-2)}{h} \\ &= \lim_{h\to 0} \frac{-2 + 3h + 2h^2 + 2}{h} \\ &= \lim_{h\to 0} \frac{3h + 2h^2}{h} \\ &= \lim_{h\to 0} \frac{h(3 + 2h)}{h} \\ &= \lim_{h\to 0} (3 + 2h) \\ &= 3 \end{aligned}$$

Step 4: Write the final answer

The gradient of the tangent to the curve $f(x) = 2x^2 - 5x$ at $x = 2$ is 3.

Worked example 6: Gradient at a point

QUESTION

Determine the gradient of $k(x) = -x^3 + 2x + 1$ at the point $x = 1$.

SOLUTION

Step 1: Write down the formula for the gradient at a point

$$\text{Gradient at a point } = \lim_{h\to 0} \frac{k\left(a+h\right) - k\left(a\right)}{h}$$

Step 2: Determine $k\left(a+h\right)$ and $k(a)$

Let $a = 1$:

$$k(x) = -x^3 + 2x + 1$$

$$\begin{aligned} k(a) &= k(1) \\ &= -(1)^3 + 2(1) + 1 \\ &= -1 + 2 + 1 \\ &= 2 \end{aligned}$$

$$\begin{aligned} k\left(a+h\right) &= k\left(1+h\right) \\ &= -\left(1+h\right)^3 + 2\left(1+h\right) + 1 \\ &= -\left(1 + 3h + 3h^2 + h^3\right) + 2 + 2h + 1 \\ &= -1 - 3h - 3h^2 - h^3 + 2 + 2h + 1 \\ &= 2 - h - 3h^2 - h^3 \end{aligned}$$

Step 3: Substitute into the formula and simplify

$$\begin{aligned} \lim_{h\to 0} \frac{k\left(a+h\right) - k\left(a\right)}{h} &= \lim_{h\to 0} \frac{k\left(1+h\right) - k\left(1\right)}{h} \\ &= \lim_{h\to 0} \frac{\left(2 - h - 3h^2 - h^3\right) - 2}{h} \\ &= \lim_{h\to 0} \frac{-h - 3h^2 - h^3}{h} \\ &= \lim_{h\to 0} \frac{h\left(-1 - 3h - h^2\right)}{h} \\ &= \lim_{h\to 0} \left(-1 - 3h - h^2\right) \\ &= -1 \end{aligned}$$

Step 4: Write the final answer

The gradient of $k(x) = -x^3 + 2x + 1$ at $x = 1$ is -1.

Exercise 6 – 2: Gradient at a point

1. Given: $f(x) = -x^2 + 7$
   a) Find the average gradient of function f, between $x = -1$ and $x = 3$.
   b) Illustrate this with a graph.
   c) Find the gradient of f at the point $x = 3$ and illustrate this on your graph.
2. Determine the gradient of the tangent to g if $g(x) = \frac{3}{x}$ $(x \neq 0)$ at $x = a$.
3. Determine the equation of the tangent to $H(x) = x^2 + 3x$ at $x = -1$.
4. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 28VX 2. 28VY 3. 28VZ

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6.2 Differentiation from first principles EMCH6

We know that the gradient of the tangent to a curve with equation $y = f(x)$ at $x = a$ can be determine using the formula:

$$\text{Gradient at a point } = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

We can use this formula to determine an expression that describes the gradient of the graph (or the gradient of the tangent to the graph) at any point on the graph. This expression (or gradient function) is called the derivative.

DEFINITION: *Derivative*

The derivative of a function $f(x)$ is written as $f'(x)$ and is defined by:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

DEFINITION: *Differentiation*

The process of determining the derivative of a given function.

This method is called differentiation from first principles or using the definition.

Worked example 7: Differentiation from first principles

QUESTION

Calculate the derivative of $g(x) = 2x - 3$ from first principles.

SOLUTION

Step 1: Write down the formula for finding the derivative using first principles

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

Step 2: Determine $g(x+h)$

$$g(x) = 2x - 3$$

$$\begin{aligned} g(x+h) &= 2(x+h) - 3 \\ &= 2x + 2h - 3 \end{aligned}$$

Step 3: Substitute into the formula and simplify

$$\begin{aligned} g'(x) &= \lim_{h \to 0} \frac{2x + 2h - 3 - (2x - 3)}{h} \\ &= \lim_{h \to 0} \frac{2h}{h} \\ &= \lim_{h \to 0} 2 \\ &= 2 \end{aligned}$$

Step 4: Write the final answer

The derivative $g'(x) = 2$.

Notation

There are a few different notations used to refer to derivatives. It is very important that you learn to identify these different ways of denoting the derivative and that you are consistent in your usage of them when answering questions.

If we use the common notation $y = f(x)$, where the dependent variable is y and the independent variable is x, then some alternative notations for the derivative are as follows:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}[f(x)] = Df(x) = D_x y$$

The symbols D and $\frac{d}{dx}$ are called differential operators because they indicate the operation of differentiation.

$\frac{dy}{dx}$ means y differentiated with respect to x. Similarly, $\frac{dp}{dx}$ means p differentiated with respect to x.

Important: $\frac{dy}{dx}$ is not a fraction and does not mean $dy \div dx$.

See video: 28W2 at www.everythingmaths.co.za

Worked example 8: Differentiation from first principles

QUESTION

1. Find the derivative of $f(x) = 4x^3$ from first principles.
2. Determine $f'(0{,}5)$ and interpret the answer.

SOLUTION

Step 1: Write down the formula for finding the derivative from first principles

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Step 2: Substitute into the formula and simplify

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{4(x+h)^3 - 4x^3}{h} \\ &= \lim_{h \to 0} \frac{4\left(x^3 + 3x^2h + 3xh^2 + h^3\right) - 4x^3}{h} \\ &= \lim_{h \to 0} \frac{4x^3 + 12x^2h + 12xh^2 + 4h^3 - 4x^3}{h} \\ &= \lim_{h \to 0} \frac{12x^2h + 12xh^2 + 4h^3}{h} \\ &= \lim_{h \to 0} \frac{h\left(12x^2 + 12xh + 4h^2\right)}{h} \\ &= \lim_{h \to 0} (12x^2 + 12xh + 4h^2) \\ &= 12x^2 \end{aligned}$$

Step 3: Calculate $f'(0{,}5)$ and interpret the answer

$$\begin{aligned} f'(x) &= 12x^2 \\ \therefore f'(0{,}5) &= 12(0{,}5)^2 \\ &= 12\left(\frac{1}{4}\right) \\ &= 3 \end{aligned}$$

- The derivative of $f(x)$ at $x = 0{,}5$ is 3.
- The gradient of the function f at $x = 0{,}5$ is equal to 3.
- The gradient of the tangent to $f(x)$ at $x = 0{,}5$ is equal to 3.

Worked example 9: Differentiation from first principles

QUESTION

Calculate $\frac{dp}{dx}$ from first principles if $p(x) = -\frac{2}{x}$.

SOLUTION

Step 1: Write down the formula for finding the derivative using first principles

$$\frac{dp}{dx} = \lim_{h \to 0} \frac{p(x+h) - p(x)}{h}$$

Step 2: Substitute into the formula and simplify

$$\frac{dp}{dx} = \lim_{h \to 0} \frac{-\frac{2}{x+h} - \left(-\frac{2}{x}\right)}{h}$$

It is sometimes easier to write the right-hand side of the equation as:

$$\begin{aligned}
\frac{dp}{dx} &= \lim_{h \to 0} \frac{1}{h}\left(\frac{-2}{x+h} + \frac{2}{x}\right) \\
&= \lim_{h \to 0} \frac{1}{h}\left(\frac{-2x + 2(x+h)}{x(x+h)}\right) \\
&= \lim_{h \to 0} \frac{1}{h}\left(\frac{-2x + 2x + 2h}{x(x+h)}\right) \\
&= \lim_{h \to 0} \frac{1}{h}\left(\frac{2h}{x^2 + xh}\right) \\
&= \lim_{h \to 0} \frac{2}{x^2 + xh} \\
&= \frac{2}{x^2}
\end{aligned}$$

Notice: even though h remains in the denominator, we can take the limit since it does not result in division by 0.

Step 3: Write the final answer

$$\frac{dp}{dx} = \frac{2}{x^2}$$

Worked example 10: Differentiation from first principles

QUESTION

Differentiate $g(x) = \frac{1}{4}$ from first principles and interpret the answer.

SOLUTION

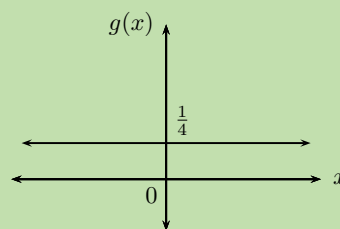
Step 1: Write down the formula for finding the derivative from first principles

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

Step 2: Substitute into the formula and simplify

$$\begin{aligned} g'(x) &= \lim_{h \to 0} \frac{\frac{1}{4} - \frac{1}{4}}{h} \\ &= \lim_{h \to 0} \frac{0}{h} \\ &= \lim_{h \to 0} 0 \\ &= 0 \end{aligned}$$

Step 3: Interpret the answer

The gradient of $g(x)$ is equal to 0 at any point on the graph. The derivative of this constant function is equal to 0.

Exercise 6 – 3: Differentiation from first principles

1. Given: $g(x) = -x^2$
   a) Determine $\frac{g(x+h)-g(x)}{h}$.
   b) Hence, determine $\lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$.
   c) Explain the meaning of your answer in (b).
2. Find the derivative of $f(x) = -2x^2 + 3x + 1$ using first principles.
3. Determine the derivative of $f(x) = \frac{1}{x-2}$ using first principles.
4. Determine $g'(3)$ from first principles if $g(x) = -5x^2$.
5. If $p(x) = 4x(x-1)$, determine $p'(x)$ using first principles.
6. Find the derivative of $k(x) = 10x^3$ using first principles.
7. Differentiate $f(x) = x^n$ using first principles.
   (Hint: Use Pascal's triangle)
8. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 28W3 1b. 28W4 1c. 28W5 2. 28W6 3. 28W7 4. 28W8
5. 28W9 6. 28WB 7. 28WC

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6.3 Rules for differentiation EMCH7

Determining the derivative of a function from first principles requires a long calculation and it is easy to make mistakes. However, we can use this method of finding the derivative from first principles to obtain rules which make finding the derivative of a function much simpler.

Investigation: Rules for differentiation

1. Differentiate the following from first principles:

a) $f(x) = x$

b) $f(x) = -4x$

c) $f(x) = x^2$

d) $f(x) = 3x^2$

e) $f(x) = -x^3$

f) $f(x) = 2x^3$

g) $f(x) = \frac{1}{x}$

h) $f(x) = -\frac{2}{x}$

2. Complete the table:

$f(x)$	$f'(x)$
x	
$-4x$	
x^2	
$3x^2$	
$-x^3$	
$2x^3$	
$\frac{1}{x}$	
$-\frac{2}{x}$	

3. Can you identify a pattern for determining the derivative?

Rules for differentiation

- General rule for differentiation:

$$\frac{d}{dx}[x^n] = nx^{n-1}, \text{ where } n \in \mathbb{R} \text{ and } n \neq 0.$$

- The derivative of a constant is equal to zero.

$$\frac{d}{dx}[k] = 0$$

- The derivative of a constant multiplied by a function is equal to the constant multiplied by the derivative of the function.

$$\frac{d}{dx}[k \cdot f(x)] = k\frac{d}{dx}[f(x)]$$

- The derivative of a sum is equal to the sum of the derivatives.

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$$

- The derivative of a difference is equal to the difference of the derivatives.

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)]$$

See video: 28WD at www.everythingmaths.co.za

Worked example 11: Rules of differentiation

QUESTION

Use the rules of differentiation to find the derivative of each of the following:

1. $y = 3x^5$
2. $p = \frac{1}{4}q^2$
3. $f(x) = 60$
4. $y = 12x^3 + 7x$
5. $m = \frac{3}{2}n^4 - 1$

SOLUTION

Step 1: Apply the appropriate rules to determine the derivative

1. $\frac{dy}{dx} = 3\left(5x^4\right) = 15x^4$
2. $\frac{dp}{dq} = \frac{1}{4}\left(2q\right) = \frac{1}{2}q$
3. $f'(x) = 0$
4. $\frac{dy}{dx} = 12\left(3x^2\right) + 7 = 36x^2 + 7$
5. $\frac{dm}{dn} = \frac{3}{2}\left(4n^3\right) - 0 = 6n^3$

Worked example 12: Rules of differentiation

QUESTION

Differentiate the following with respect to t:

1. $g(t) = 4\left(t+1\right)^2\left(t-3\right)$
2. $k(t) = \frac{(t+2)^3}{\sqrt{t}}$

SOLUTION

Step 1: Expand the expression and apply the rules of differentiation

We have not learnt a rule for differentiating a product, therefore we must expand the brackets and simplify before we can determine the derivative:

$$\begin{aligned} g(t) &= 4\left(t+1\right)^2\left(t-3\right) \\ &= 4\left(t^2+2t+1\right)\left(t-3\right) \\ &= 4\left(t^3+2t^2+t-3t^2-6t-3\right) \\ &= 4\left(t^3-t^2-5t-3\right) \\ &= 4t^3-4t^2-20t-12 \end{aligned}$$

$$\begin{aligned} \therefore g'(t) &= 4\left(3t^2\right)-4\left(2t\right)-20-0 \\ &= 12t^2-8t-20 \end{aligned}$$

Step 2: Expand the expression and apply the rules of differentiation

We have not learnt a rule for differentiating a quotient, therefore we must first simplify the expression and then we can differentiate:

$$\begin{aligned} k(t) &= \frac{(t+2)^3}{\sqrt{t}} \\ &= \frac{(t+2)(t^2+4t+4)}{\sqrt{t}} \\ &= \frac{t^3+6t^2+12t+8}{t^{\frac{1}{2}}} \\ &= t^{-\frac{1}{2}}\left(t^3+6t^2+12t+8\right) \\ &= t^{\frac{5}{2}}+6t^{\frac{3}{2}}+12t^{\frac{1}{2}}+8t^{-\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} \therefore g'(t) &= \frac{5}{2}t^{\frac{3}{2}}+6\left(\frac{3}{2}t^{\frac{1}{2}}\right)+12\left(\frac{1}{2}t^{-\frac{1}{2}}\right)+8\left(-\frac{1}{2}t^{-\frac{3}{2}}\right) \\ &= \frac{5}{2}t^{\frac{3}{2}}+9t^{\frac{1}{2}}+6t^{-\frac{1}{2}}-4t^{-\frac{3}{2}} \end{aligned}$$

Important: always write the final answer with positive exponents.

$$g'(t) = \frac{5}{2}t^{\frac{3}{2}}+9t^{\frac{1}{2}}+\frac{6}{t^{\frac{1}{2}}}-\frac{4}{t^{\frac{3}{2}}}$$

When to use the rules for differentiation:

- If the question does not specify how we must determine the derivative, then we use the rules for differentiation.

When to differentiate using first principles:

- If the question specifically states to use first principles.
- If we are required to differentiate using the definition of a derivative, then we use first principles.

Exercise 6 – 4: Rules for differentiation

1. Differentiate the following:

a) $y = 3x^2$

b) $f(x) = 25x$

c) $k(x) = -30$

d) $y = -4x^5 + 2$

e) $g(x) = 16x^{-2}$

f) $y = 10(7-3)$

g) $q(x) = x^4 - 6x^2 - 1$

h) $y = x^2 + x + 4$

i) $f(x) = \frac{1}{3}x^3 - x^2 + \frac{2}{5}$

j) $y = 3x^{\frac{3}{2}} - 4x + 20$

k) $g(x) = x(x+2) + 5x$

l) $p(x) = 200[x^3 - \frac{1}{2}x^2 + \frac{1}{5}x - 40]$

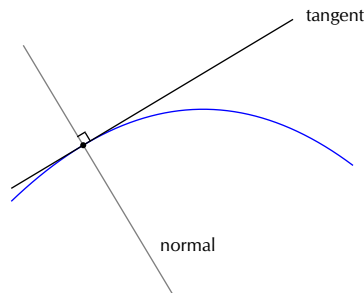
m) $y = 14(x-1)\left[\frac{1}{2} + x^2\right]$

2. Find $f'(x)$ if $f(x) = \frac{x^2-5x+6}{x-2}$.
3. Find $f'(y)$ if $f(y) = \sqrt{y}$.
4. Find $f'(z)$ if $f(z) = (z-1)(z+1)$.
5. Determine $\frac{dy}{dx}$ if $y = \frac{x^3+2\sqrt{x}-3}{x}$.
6. Determine the derivative of $y = \sqrt{x^3} + \frac{1}{3x^3}$.
7. Find $D_x \left[x^{\frac{3}{2}} - \frac{3}{x^{\frac{1}{2}}}\right]^2$.
8. Find $\frac{dy}{dx}$ if $x = 2y + 3$.
9. Determine $f'(\theta)$ if $f(\theta) = 2(\theta^{\frac{3}{2}} - 3\theta^{-\frac{1}{2}})^2$.
10. Find $\frac{dp}{dt}$ if $p(t) = \frac{(t+1)^3}{\sqrt{t}}$.
11. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 28WF	1b. 28WG	1c. 28WH	1d. 28WJ	1e. 28WK	1f. 28WM
1g. 28WN	1h. 28WP	1i. 28WQ	1j. 28WR	1k. 28WS	1l. 28WT
1m. 28WV	2. 28WW	3. 28WX	4. 28WY	5. 28WZ	6. 28X2
7. 28X3	8. 28X4	9. 28X5	10. 28X6		

6.4 Equation of a tangent to a curve EMCH8

At a given point on a curve, the gradient of the curve is equal to the gradient of the tangent to the curve.

The derivative (or gradient function) describes the gradient of a curve at any point on the curve. Similarly, it also describes the gradient of a tangent to a curve at any point on the curve.

To determine the equation of a tangent to a curve:

1. Find the derivative using the rules of differentiation.
2. Substitute the x-coordinate of the given point into the derivative to calculate the gradient of the tangent.
3. Substitute the gradient of the tangent and the coordinates of the given point into an appropriate form of the straight line equation.
4. Make y the subject of the formula.

The normal to a curve is the line perpendicular to the tangent to the curve at a given point.

$$m_{\text{tangent}} \times m_{\text{normal}} = -1$$

Worked example 13: Finding the equation of a tangent to a curve

QUESTION

Find the equation of the tangent to the curve $y = 3x^2$ at the point $(1; 3)$. Sketch the curve and the tangent.

SOLUTION

Step 1: Find the derivative

Use the rules of differentiation:

$$y = 3x^2$$

$$\therefore \frac{dy}{dx} = 3(2x)$$
$$= 6x$$

Step 2: Calculate the gradient of the tangent

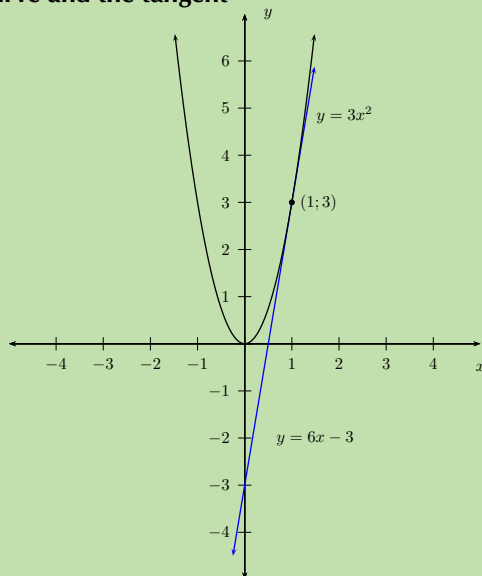
To determine the gradient of the tangent at the point $(1; 3)$, we substitute the x-value into the equation for the derivative.

$$\frac{dy}{dx} = 6x$$
$$\therefore m = 6(1)$$
$$= 6$$

Step 3: Determine the equation of the tangent

Substitute the gradient of the tangent and the coordinates of the given point into the gradient-point form of the straight line equation.

$$y - y_1 = m(x - x_1)$$
$$y - 3 = 6(x - 1)$$
$$y = 6x - 6 + 3$$
$$y = 6x - 3$$

Step 4: Sketch the curve and the tangent

Worked example 14: Finding the equation of a tangent to a curve

QUESTION

Given $g(x) = (x + 2)(2x + 1)^2$, determine the equation of the tangent to the curve at $x = -1$.

SOLUTION

Step 1: Determine the y-coordinate of the point

$$\begin{aligned} g(x) &= (x + 2)(2x + 1)^2 \\ g(-1) &= (-1 + 2)[2(-1) + 1]^2 \\ &= (1)(-1)^2 \\ &= 1 \end{aligned}$$

Therefore the tangent to the curve passes through the point $(-1; 1)$.

Step 2: Expand and simplify the given function

$$\begin{aligned} g(x) &= (x + 2)(2x + 1)^2 \\ &= (x + 2)(4x^2 + 4x + 1) \\ &= 4x^3 + 4x^2 + x + 8x^2 + 8x + 2 \\ &= 4x^3 + 12x^2 + 9x + 2 \end{aligned}$$

Step 3: Find the derivative

$$\begin{aligned} g'(x) &= 4(3x^2) + 12(2x) + 9 + 0 \\ &= 12x^2 + 24x + 9 \end{aligned}$$

Step 4: Calculate the gradient of the tangent

Substitute $x = -1$ into the equation for $g'(x)$:

$$\begin{aligned} g'(-1) &= 12(-1)^2 + 24(-1) + 9 \\ \therefore m &= 12 - 24 + 9 \\ &= -3 \end{aligned}$$

Step 5: Determine the equation of the tangent

Substitute the gradient of the tangent and the coordinates of the point into the gradient-point form of the straight line equation.

$$\begin{aligned} y - y_1 &= m\,(x - x_1) \\ y - 1 &= -3\,(x - (-1)) \\ y &= -3x - 3 + 1 \\ y &= -3x - 2 \end{aligned}$$

Worked example 15: Finding the equation of a normal to a curve

QUESTION

1. Determine the equation of the normal to the curve $xy = -4$ at $(-1;4)$.
2. Draw a rough sketch.

SOLUTION

Step 1: Find the derivative

Make y the subject of the formula and differentiate with respect to x:

$$\begin{aligned} y &= -\frac{4}{x} \\ &= -4x^{-1} \end{aligned}$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= -4\left(-1x^{-2}\right) \\ &= 4x^{-2} \\ &= \frac{4}{x^2} \end{aligned}$$

Step 2: Calculate the gradient of the normal at $(-1;4)$

First determine the gradient of the tangent at the given point:

$$\begin{aligned} \frac{dy}{dx} &= \frac{4}{(-1)^2} \\ \therefore m &= 4 \end{aligned}$$

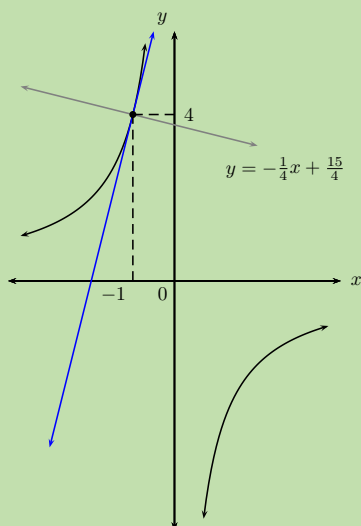
Use the gradient of the tangent to calculate the gradient of the normal:

$$\begin{aligned} m_{\text{tangent}} \times m_{\text{normal}} &= -1 \\ 4 \times m_{\text{normal}} &= -1 \\ \therefore m_{\text{normal}} &= -\frac{1}{4} \end{aligned}$$

Step 3: Find the equation of the normal

Substitute the gradient of the normal and the coordinates of the given point into the gradient-point form of the straight line equation.

$$\begin{aligned} y - y_1 &= m\left(x - x_1\right) \\ y - 4 &= -\frac{1}{4}\left(x - (-1)\right) \\ y &= -\frac{1}{4}x - \frac{1}{4} + 4 \\ y &= -\frac{1}{4}x + \frac{15}{4} \end{aligned}$$

Step 4: Draw a rough sketch

Exercise 6 – 5: Equation of a tangent to a curve

1. Determine the equation of the tangent to the curve defined by $F(x) = x^3 + 2x^2 - 7x + 1$ at $x = 2$.
2. Determine the point where the gradient of the tangent to the curve:
   a) $f(x) = 1 - 3x^2$ is equal to 5.
   b) $g(x) = \frac{1}{3}x^2 + 2x + 1$ is equal to 0.
3. Determine the point(s) on the curve $f(x) = (2x - 1)^2$ where the tangent is:
   a) parallel to the line $y = 4x - 2$.
   b) perpendicular to the line $2y + x - 4 = 0$.
4. Given the function $f\colon y = -x^2 + 4x - 3$.
   a) Draw a graph of f, indicating all intercepts and turning points.
   b) Find the equations of the tangents to f at:
      i. the y-intercept of f.
      ii. the turning point of f.
      iii. the point where $x = 4{,}25$.
   c) Draw the three tangents above on your graph of f.
   d) Write down all observations about the three tangents to f.
5. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 28X7   2a. 28X8   2b. 28X9   3. 28XB   4. 28XC

6.5 Second derivative EMCH9

The second derivative of a function is the derivative of the first derivative and it indicates the change in gradient of the original function. The sign of the second derivative tells us if the gradient of the original function is increasing, decreasing or remaining constant.

To determine the second derivative of the function $f(x)$, we differentiate $f'(x)$ using the rules for differentiation.

$$f''(x) = \frac{d}{dx}[f'(x)]$$

We also use the following notation for determining the second derivative of y:

$$y'' = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d^2y}{dx^2}$$

Worked example 16: Finding the second derivative

QUESTION

Calculate the second derivative for each of the following:

1. $k(x) = 2x^3 - 4x^2 + 9$
2. $y = \frac{3}{x}$

SOLUTION

1.

$$\begin{aligned} k'(x) &= 2(3x^2) - 4(2x) + 0 \\ &= 6x^2 - 8x \end{aligned}$$

$$\begin{aligned} k''(x) &= 6(2x) - 8 \\ &= 12x - 8 \end{aligned}$$

2.

$$y = 3x^{-1}$$

$$\begin{aligned} \frac{dy}{dx} &= 3(-1x^{-2}) \\ &= -3x^{-2} \\ &= -\frac{3}{x^2} \end{aligned}$$

$$\begin{aligned} \frac{d^2y}{dx^2} &= -3(-2x^{-3}) \\ &= \frac{6}{x^3} \end{aligned}$$

Exercise 6 – 6: Second derivative

1. Calculate the second derivative for each of the following:

   a) $g(x) = 5x^2$
   b) $y = 8x^3 - 7x$
   c) $f(x) = x(x-6) + 10$
   d) $y = x^5 - x^3 + x - 1$
   e) $k(x) = (x^2+1)(x-1)$
   f) $p(x) = -\frac{10}{x^2}$
   g) $q(x) = \sqrt{x} + 5x^2$

2. Find the first and second derivatives of $f(x) = 5x(2x+3)$.
3. Find $\frac{d^2}{dx^2}\left[6\sqrt[3]{x^2}\right]$.
4. Given the function $g: \quad y = (1-2x)^3$.
   a) Determine g' and g''.
   b) What type of function is:
      i. g'
      ii. g''
   c) Find the value of $g''\left(\frac{1}{2}\right)$.
   d) What do you observe about the degree (highest power) of each of the derived functions?
5. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 28XD 1b. 28XF 1c. 28XG 1d. 28XH 1e. 28XJ 1f. 28XK
1g. 28XM 2. 28XN 3. 28XP 4. 28XQ

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6.6 Sketching graphs EMCHB

Functions of the form $y = ax^3 + bx^2 + cx + d$ EMCHC

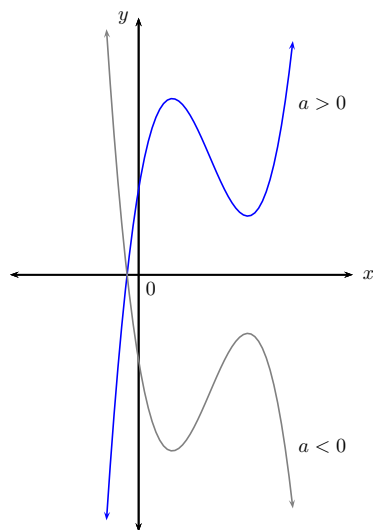
Investigation: The effects of a on a cubic function

Complete the table below and plot the graphs of $f(x)$ and $g(x)$ on the same system of axes.

Be careful to choose a suitable scale for the y-axis.

$f(x) = 2x^3 - 5x^2 - 14x + 8$ $\qquad$ $g(x) = -2x^3 + 5x^2 + 14x - 8$

x	-3	-2	-1	0	1	2	3	4	5
$f(x)$							-25		
$g(x)$							25		

The effects of a on a cubic graph

Intercepts

EMCHD

Investigation: Number of intercepts

Complete the table:

	$y = x + 1$	$y = x^2 - x - 6$	$y = x^3 + x^2 - 26x + 24$
Degree of function			
Type of function			
Factorised form			
No. of x-intercepts			
No. of y-intercepts			

Worked example 17: Determining the intercepts

QUESTION

Given $f(x) = -x^3 + 4x^2 + x - 4$, find the x- and y-intercepts.

SOLUTION

Step 1: Determine the y-intercept

The y-intercept is obtained by letting $x = 0$:

$$\begin{aligned} y &= -(0)^3 + 4(0)^2 + (0) - 4 \\ &= -4 \end{aligned}$$

This gives the point $(0; -4)$.

Step 2: Use the factor theorem to factorise the expression

We use the factor theorem to find a factor of $f(x)$ by trial and error:

$$\begin{aligned} f(x) &= -x^3 + 4x^2 + x - 4 \\ f(1) &= -(1)^3 + 4(1)^2 + (1) - 4 \\ &= 0 \\ \therefore (x-1) & \text{ is a factor of } f(x) \end{aligned}$$

Factorise further by inspection:

$$\begin{aligned} f(x) &= (x-1)(-x^2 + 3x + 4) \\ &= -(x-1)(x^2 - 3x - 4) \\ &= -(x-1)(x+1)(x-4) \end{aligned}$$

The x-intercepts are obtained by letting $f(x) = 0$:

$$\begin{aligned} 0 &= -(x-1)(x+1)(x-4) \\ \therefore x = -1, x = 1 & \text{ or } x = 4 \end{aligned}$$

This gives the points $(-1;0)$, $(1;0)$ and $(4;0)$.

Exercise 6 – 7: Intercepts

1. Given the function $f(x) = x^3 + x^2 - 10x + 8$.
   a) Determine the x- and y-intercepts of $f(x)$.
   b) Draw a rough sketch of the graph.
   c) Is the function increasing or decreasing at $x = -5$?
2. Determine the x- and y-intercepts for each of the following:
   a) $y = -x^3 - 5x^2 + 9x + 45$
   b) $y = x^3 - \frac{5}{4}x^2 - \frac{7}{4}x + \frac{1}{2}$
   c) $y = x^3 - x^2 - 12x + 12$
   d) $y = x^3 - 16x$
   e) $y = x^3 - 5x^2 + 6$
3. Determine all intercepts for $g(x) = x^3 + 3x^2 - 10x$ and draw a rough sketch of the graph.
4. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 28XR 2a. 28XS 2b. 28XT 2c. 28XV 2d. 28XW 2e. 28XX
3. 28XY

Investigation:

1. Complete the table below for the quadratic function $f(x)$:
$$f(x) = x^2 + 2x + 1$$
$$f'(x) = \ldots\ldots\ldots$$

x-value	-3	-1	0	1	3
Gradient of f					
Sign of gradient					
Increasing function ($\nearrow$) Decreasing function ($\searrow$) Maximum TP ($\cap$) Minimum TP ($\cup$)					

2. Use the table to draw a rough sketch of the graph of $f(x)$.
3. Solve for x if $f'(x) = 0$.
4. Indicate solutions to $f'(x) = 0$ on the graph.
5. Complete the table below for the cubic function $g(x)$:
$$g(x) = 2x^3 + 3x^2 - 12x$$
$$g'(x) = \ldots\ldots\ldots$$

x-value	-3	-2	0	1	3
Gradient of g					
Sign of gradient					
Increasing function ($\nearrow$) Decreasing function ($\searrow$) Maximum TP ($\cap$) Minimum TP ($\cup$)					

6. Use the table to draw a rough sketch of the graph of $g(x)$.
7. Solve for x if $g'(x) = 0$.
8. Indicate solutions to $g'(x) = 0$ on the graph.
9. Complete the following sentence:
The derivative describes the ...... of a tangent to a curve at a given point and we have seen that the ...... of a curve at its stationary point(s) is equal to .... Therefore, we can use ...... as a tool for finding the stationary points of the graphs of quadratic and cubic functions.

To determine the coordinates of the stationary point(s) of $f(x)$:

- Determine the derivative $f'(x)$.
- Let $f'(x) = 0$ and solve for the x-coordinate(s) of the stationary point(s).
- Substitute value(s) of x into $f(x)$ to calculate the y-coordinate(s) of the stationary point(s).

Worked example 18: Finding stationary points

QUESTION

Calculate the stationary points of the graph of $p(x) = x^3 - 6x^2 + 9x - 4$.

SOLUTION

Step 1: Determine the derivative of $p(x)$

Using the rules of differentiation we get:

$$p'(x) = 3x^2 - 12x + 9$$

Step 2: Let $p'(x) = 0$ and solve for x

$$\begin{aligned} 3x^2 - 12x + 9 &= 0 \\ x^2 - 4x + 3 &= 0 \\ (x-3)(x-1) &= 0 \\ \therefore x = 1 \text{ or } x &= 3 \end{aligned}$$

Therefore, the x-coordinates of the turning points are $x = 1$ and $x = 3$.

Step 3: Substitute the x-values into $p(x)$

We use the x-coordinates to calculate the corresponding y-coordinates of the stationary points.

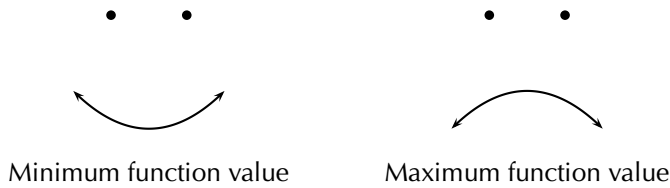
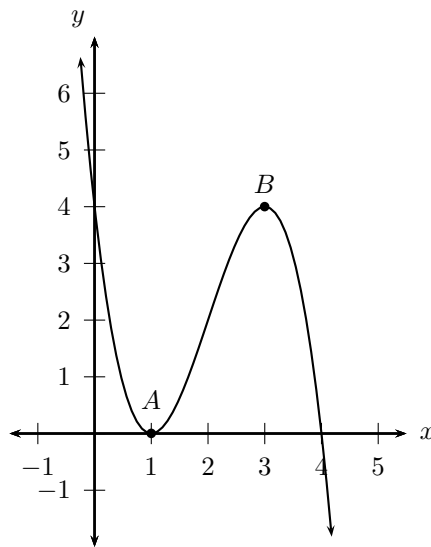
$$\begin{aligned} p(1) &= (1)^3 - 6(1)^2 + 9(1) - 4 \\ &= 1 - 6 + 9 - 4 \\ &= 0 \end{aligned}$$

$$\begin{aligned} p(3) &= (3)^3 - 6(3)^2 + 9(3) - 4 \\ &= 27 - 54 + 27 - 4 \\ &= -4 \end{aligned}$$

Step 4: Write final answer

The turning points of the graph of $p(x) = x^3 - 6x^2 + 9x - 4$ are $(1; 0)$ and $(3; -4)$.

Local maximum and local minimum

We have seen that the graph of a quadratic function can have either a minimum turning point ("smile") or a maximum turning point ("frown").

For cubic functions, we refer to the turning (or stationary) points of the graph as local minimum or local maximum turning points. The diagram below shows local minimum turning point $A(1; 0)$ and local maximum turning point $B(3; 4)$. These points are described as a local (or relative) minimum and a local maximum because there are other points on the graph with lower and higher function values.

Exercise 6 – 8: Stationary points

1. Use differentiation to determine the stationary point(s) for $g(x) = -x^2 + 5x - 6$.
2. Determine the x-values of the stationary points for $f(x) = -\frac{1}{3}x^3 + \frac{1}{2}x^2 + 6x + 5$.
3. Find the coordinates of the stationary points of the following functions using the rules of differentiation:
   a) $y = (x - 1)^3$
   b) $y = x^3 - 5x^2 + 1$
   c) $y + 7x = 1$
4. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 28XZ 2. 28Y2 3a. 28Y3 3b. 28Y4 3c. 28Y5

General method for sketching cubic graphs:

1. Consider the sign of a and determine the general shape of the graph.
2. Determine the y-intercept by letting $x = 0$.
3. Determine the x-intercepts by factorising $ax^3 + bx^2 + cx + d = 0$ and solving for x.
4. Find the x-coordinates of the turning points of the function by letting $f'(x) = 0$ and solving for x.
5. Determine the y-coordinates of the turning points by substituting the x-values into $f(x)$.
6. Plot the points and draw a smooth curve.

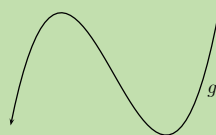
Worked example 19: Sketching cubic graphs

QUESTION

Sketch the graph of $g(x) = x^3 - 3x^2 - 4x$.

SOLUTION

Step 1: Determine the shape of the graph

The coefficient of the x^3 term is greater than zero, therefore the graph will have the following shape:

Step 2: Determine the intercepts

The y-intercept is obtained by letting $x = 0$:

$$\begin{aligned} g(0) &= (0)^3 - 3(0)^2 - 4(0) \\ &= 0 \end{aligned}$$

This gives the point $(0;0)$.

The x-intercept is obtained by letting $g(x) = 0$ and solving for x:

$$\begin{aligned} 0 &= x^3 - 3x^2 - 4x \\ &= x(x^2 - 3x - 4) \\ &= x(x-4)(x+1) \\ \therefore x = -1,\ x &= 0 \text{ or } x = 4 \end{aligned}$$

This gives the points $(-1;0)$, $(0;0)$ and $(4;0)$.

Step 3: Calculate the stationary points

Find the x-coordinates of the stationary points by setting $g'(x) = 0$:

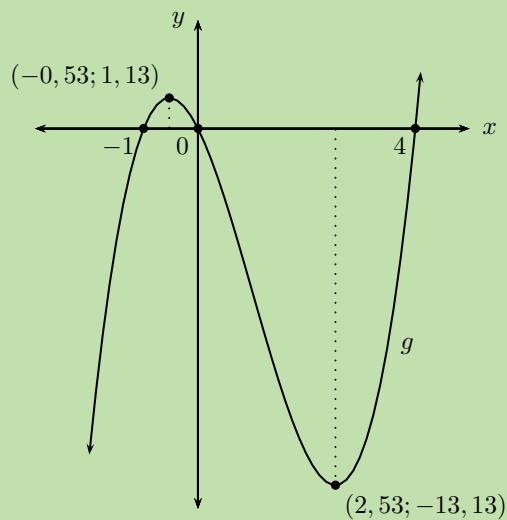
$$\begin{aligned} g'(x) &= 3x^2 - 6x - 4 \\ 0 &= 3x^2 - 6x - 4 \\ \text{Using the quadratic formula } x &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(3)(-4)}}{2(3)} \\ &= \frac{6 \pm \sqrt{36+48}}{6} \\ \therefore x = 2{,}53 &\text{ or } x = -0{,}53 \end{aligned}$$

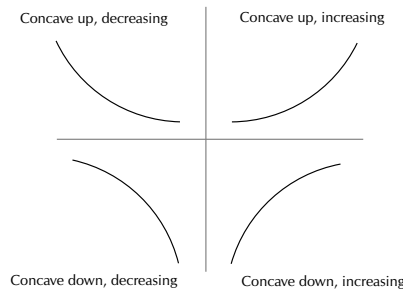
Substitute these x-coordinates into $g(x)$ to determine the corresponding y-coordinates:

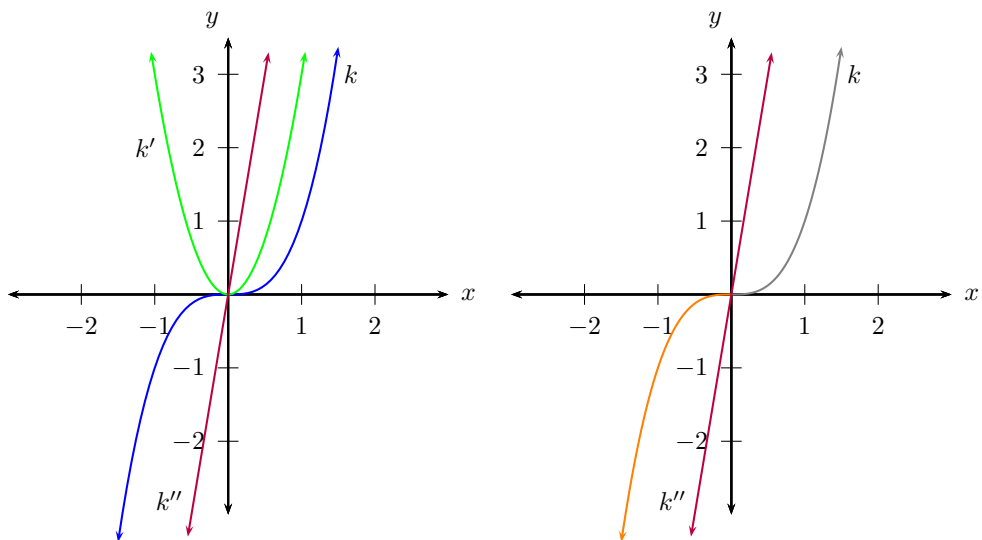
$$\begin{aligned} g(x) &= (2{,}53)^3 - 3(2{,}53)^2 - 4(2{,}53) \\ &= -13{,}13 \end{aligned}$$

$$\begin{aligned} g(x) &= (-0{,}53)^3 - 3(-0{,}53)^2 - 4(-0{,}53) \\ &= 1{,}13 \end{aligned}$$

Therefore, the stationary points are $(2{,}53; -13{,}13)$ and $(-0{,}53; 1{,}13)$.

Step 4: Draw a neat sketch

Concavity

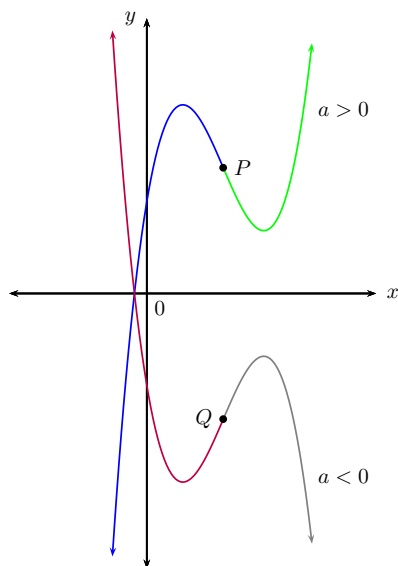
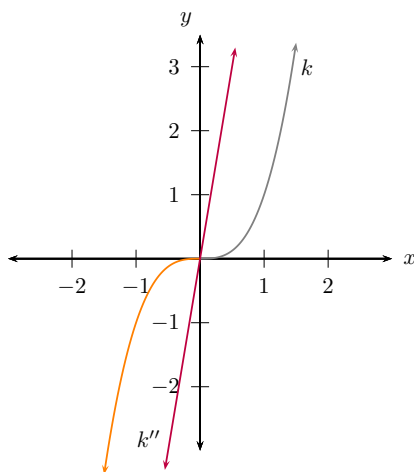
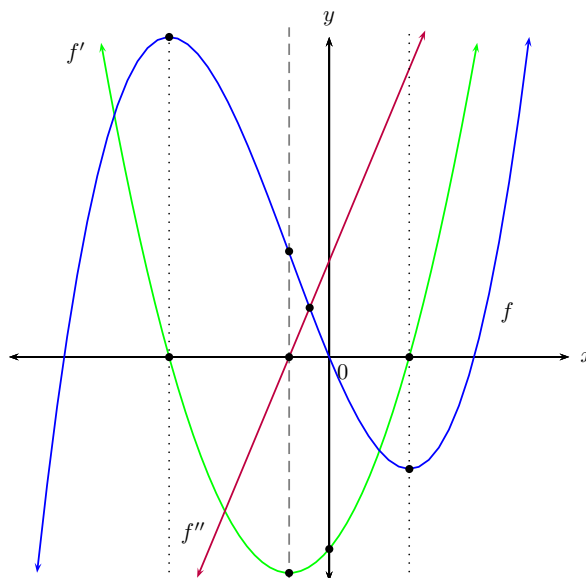
Concavity indicates whether the gradient of a curve is increasing, decreasing or stationary.

- Concave up: the gradient of the curve increases as x increases.
- Concave down: the gradient of the curve decreases as x increases.
- Zero concavity: the gradient of the curve is constant.

The diagram below shows the graph of the cubic function $k(x) = x^3$. The first derivative of $k(x)$ is a quadratic function, $k'(x) = 3x^2$ and the second derivative is a linear function, $k''(x) = 6x$.

Notice the following:

- $k''(x) > 0$, the graph is concave up.
- $k''(x) < 0$, the graph is concave down.
- $k''(x) = 0$, change in concavity (point of inflection).

Points of inflection

This is the point where the concavity of a curve changes, as shown in the diagram below. If $a < 0$, then the concavity changes from concave up (purple) to concave down (grey) and if $a > 0$, concavity changes from concave down (blue) to concave up (green). Unlike a turning point, the gradient of the curve on the left-hand side of

an inflection point (P and Q) has the same sign as the gradient of the curve on the right-hand side.

A graph has a horizontal point of inflection where the derivative is zero but the sign of the gradient of the curve does not change. That means the graph (shown below) will continue to increase or decrease after the stationary point.

In the example above, the equation $k'(x) = 3x^2$ indicates that the gradient of this curve will always be positive (except where $x = 0$). Therefore, the stationary point is a point of inflection.

f: cubic function	f': quadratic function	f'': linear function
(blue graph)	(green graph)	(red graph)
turning points $\rightarrow$	x-intercepts	
point of inflection	$\leftarrow$ turning point $\rightarrow$	x-intercept

Exercise 6 – 9: Concavity and points of inflection

Complete the following for each function:

- Determine and discuss the change in gradient of the function.
- Determine the concavity of the graph.
- Find the inflection point.
- Draw a sketch of the graph.

1. $f : y = -2x^3$
2. $g(x) = \frac{1}{8}x^3 + 1$
3. $h : x \rightarrow (x - 2)^3$
4. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 28Y6 2. 28Y7 3. 28Y8

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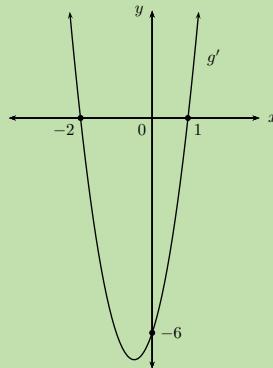
Worked example 20: Interpreting graphs

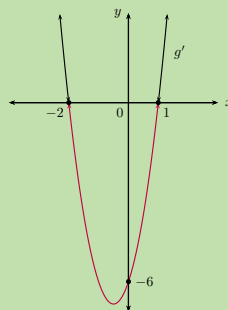
QUESTION

Consider the graph of the derivative of $g(x)$.

1. For which values of x is $g(x)$ decreasing?
2. Determine the x-coordinate(s) of the turning point(s) of $g(x)$.
3. Given that $g(x) = ax^3 + bx^2 + cx$, calculate a, b and c.

SOLUTION

Step 1: Examine the parabolic graph and interpret the given information

We know that $g'(x)$ describes the gradient of $g(x)$. To determine where the cubic function is decreasing, we must find the values of x for which $g'(x) < 0$:

$$\{x : -2 < x < 1; x \in \mathbb{R}\} \text{ or we can write } x \in (-2; 1)$$

Step 2: Determine the x-coordinate(s) of the turning point(s)

To determine the turning points of a cubic function, we let $g'(x) = 0$ and solve for the x-values. These x-values are the x-intercepts of the parabola and are indicated on the given graph:

$$x = -2 \text{ or } x = 1$$

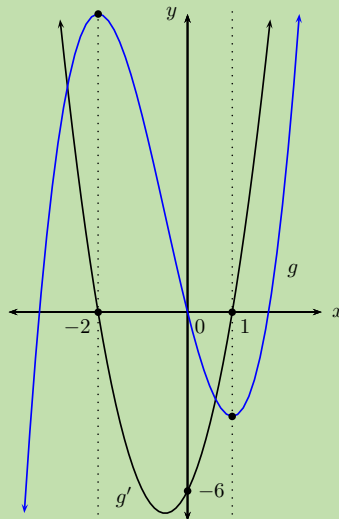
Step 3: Determine the equation of $g(x)$

$$g(x) = ax^3 + bx^2 + cx$$
$$g'(x) = 3ax^2 + 2bx + c$$

From the graph, we see that the y-intercept of $g'(x)$ is -6.

$$\therefore c = -6$$
$$g'(x) = 3ax^2 + 2bx - 6$$
$$\text{Substitute } x = -2: \quad g'(-2) = 3a(-2)^2 + 2b(-2) - 6$$
$$0 = 12a - 4b - 6 \ldots\ldots (1)$$
$$\text{Substitute } x = 1: \quad g'(1) = 3a(1)^2 + 2b(1) - 6$$
$$0 = 3a + 2b - 6 \ldots\ldots (2)$$

$$\text{Eqn. } (1) - 4 \text{ Eqn. } (2): \quad 0 = 0 - 12b + 18$$
$$\therefore b = \frac{3}{2}$$
$$\text{And } 0 = 3a + 2\left(\frac{3}{2}\right) - 6$$
$$0 = 3a - 3$$
$$\therefore a = 1$$

$$g(x) = x^3 + \frac{3}{2}x^2 - 6x$$

See video: 28VW at www.everythingmaths.co.za

Exercise 6 – 10: Mixed exercises on cubic graphs

1. Given $f(x) = x^3 + x^2 - 5x + 3$.

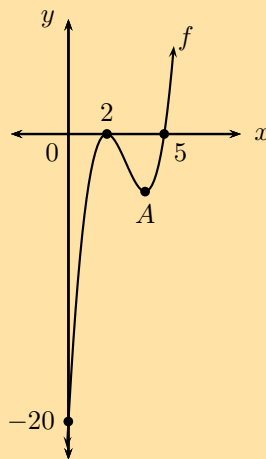
   a) Show that $(x - 1)$ is a factor of $f(x)$ and hence factorise $f(x)$.

   b) Determine the coordinates of the intercepts and the turning points.

   c) Sketch the graph.

2. a) Sketch the graph of $f(x) = -x^3 + 4x^2 + 11x - 30$. Show all the turning points and intercepts with the axes.

   b) Given $g(x) = x^3 - 4x^2 - 11x + 30$, sketch the graph of g without any further calculations. Describe the method for drawing the graph.

3. The sketch shows the graph of a cubic function, f, with a turning point at $(2; 0)$, going through $(5; 0)$ and $(0; -20)$.

   a) Find the equation of f.

   b) Find the coordinates of turning point A.

4. a) Find the intercepts and stationary point(s) of $f(x) = -\frac{1}{3}x^3 + 2$ and draw a sketch of the graph.

   b) For which values of x will:

      i. $f(x) < 0$

      ii. $f'(x) < 0$

      iii. $f''(x) < 0$

      Motivate each answer.

5. Use the information below to sketch a graph of each cubic function (do not find the equations of the functions).

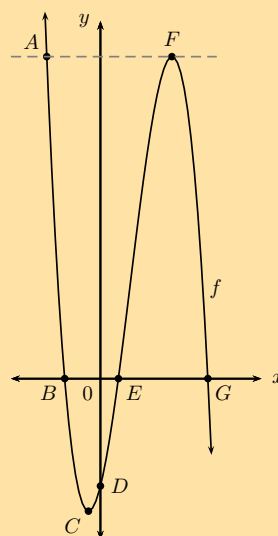
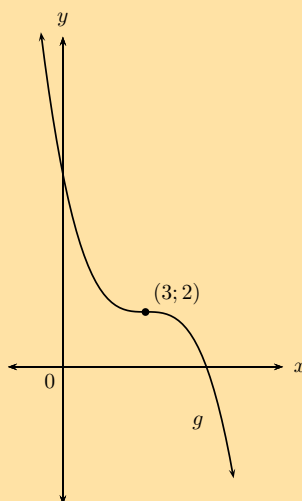
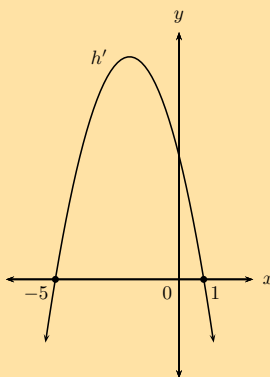
   a)

$$
\begin{aligned}
g(-6) = g(-1{,}5) = g(2) = 0 \\
g'(-4) = g'(1) = 0 \\
g'(x) > 0 \text{ for } x < -4 \text{ or } x > 1 \\
g'(x) < 0 \text{ for } -4 < x < 1
\end{aligned}
$$

b)

$$\begin{aligned} h(-3) &= 0 \\ h(0) &= 4 \\ h(-1) &= 3 \\ h'(-1) &= 0 \\ h''(-1) &= 0 \\ h'(x) &> 0 \text{ for all } x \text{ values except } x = -1 \end{aligned}$$

6. The sketch below shows the curve of $f(x) = -(x+2)(x-1)(x-6)$ with turning points at C and F. AF is parallel to the x-axis.
Determine the following:

a) length OB
b) length OE
c) length EG
d) length OD
e) coordinates of C and F
f) length AF
g) average gradient between E and F
h) the equation of the tangent to the graph at E

7. Given the graph of a cubic function with the stationary point $(3; 2)$, sketch the graph of the derivative function if it is also given that the gradient of the graph is -5 at $x = 0$.

8. The sketch below shows the graph of $h'(x)$ with x-intercepts at -5 and 1. Draw a sketch graph of $h(x)$ if $h(-5) = 2$ and $h(1) = 6$.

9. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 28Y9 2. 28YB 3. 28YC 4. 28YD 5a. 28YF 5b. 28YG
6. 28YH 7. 28YJ 8. 28YK

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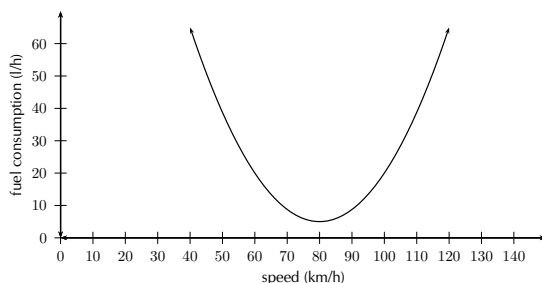
6.7 Applications of differential calculus EMCHH

Optimisation problems EMCHJ

We have seen that differential calculus can be used to determine the stationary points of functions, in order to sketch their graphs. Calculating stationary points also lends itself to the solving of problems that require some variable to be maximised or minimised. These are referred to as optimisation problems.

The fuel used by a car is defined by $f(v) = \frac{3}{80}v^2 - 6v + 245$, where v is the travelling speed in km/h.

What is the most economical speed of the car? In other words, determine the speed of the car which uses the least amount of fuel.

If we draw the graph of this function we find that the graph has a minimum. The speed at the minimum would then give the most economical speed.

We have seen that the coordinates of the turning point can be calculated by differentiating the function and finding the x-coordinate (speed in the case of the example) for which the derivative is 0.

$$f'(v) = \frac{3}{40}v - 6$$

If we set $f'(v) = 0$ we can calculate the speed that corresponds to the turning point:

$$\begin{aligned} f'(v) &= \frac{3}{40}v - 6 \\ 0 &= \frac{3}{40}v - 6 \\ v &= \frac{6 \times 40}{3} \\ &= 80 \end{aligned}$$

This means that the most economical speed is 80 km/h.

Finding the optimum point:
Let $f'(x) = 0$ and solve for x to find the optimum point.
To check whether the optimum point at $x = a$ is a local minimum or a local maximum, we find $f''(x)$:

- If $f''(a) < 0$, then the point is a local maximum.
- If $f''(a) > 0$, then the point is a local minimum.

Worked example 21: Optimisation problems

QUESTION

The sum of two positive numbers is 10. One of the numbers is multiplied by the square of the other. If each number is greater than 0, find the numbers that make this product a maximum.

Draw a graph to illustrate the answer.

SOLUTION

Step 1: Analyse the problem and formulate the equations that are required
Let the two numbers be a and b and the product be P.

$$\begin{aligned} a + b &= 10 \ldots\ldots (1) \\ P &= a \times b^2 \ldots\ldots (2) \end{aligned}$$

Make b the subject of equation (1) and substitute into equation (2):

$$\begin{aligned} P &= a(10 - a)^2 \\ &= a\left(100 - 20a + a^2\right) \\ \therefore P(a) &= 100a - 20a^2 + a^3 \end{aligned}$$

Step 2: Differentiate with respect to a

$$P'(a) = 100 - 40a + 3a^2$$

Step 3: Determine the stationary points by letting $P'(a) = 0$

We find the value of a which makes P a maximum:

$$\begin{aligned} P'(a) &= 3a^2 - 40a + 100 \\ 0 &= (3a - 10)(a - 10) \\ \therefore a = 10 &\text{ or } a = \frac{10}{3} \end{aligned}$$

Substitute into the equation (1) to solve for b:

$$\begin{aligned} \text{If } a = 10: \quad b &= 10 - 10 \\ &= 0 \ \ (\text{but } b > 0) \\ \therefore \text{no solution} \end{aligned}$$

$$\begin{aligned} \text{If } a = \frac{10}{3}: \quad b &= 10 - \frac{10}{3} \\ &= \frac{20}{3} \end{aligned}$$

Step 4: Determine the second derivative $P''(a)$

We check that the point $\left(\frac{10}{3}; \frac{20}{3}\right)$ is a local maximum by showing that $P''\left(\frac{10}{3}\right) < 0$:

$$\begin{aligned} P''(a) &= 6a - 40 \\ \therefore P''\left(\frac{10}{3}\right) &= 6\left(\frac{10}{3}\right) - 40 \\ &= 20 - 40 \\ &= -20 \end{aligned}$$

Step 5: Write the final answer

The product is maximised when the two numbers are $\frac{10}{3}$ and $\frac{20}{3}$.

Step 6: Draw a graph

To draw a rough sketch of the graph we need to calculate where the graph intersects with the axes and the maximum and minimum function values of the turning points:

Intercepts:

$$\begin{aligned} P(a) &= a^3 - 20a^2 + 100a \\ &= a(a - 10)^2 \\ \text{Let } P(a) = 0: &\ \ (0; 0) \text{ and } (10; 0) \end{aligned}$$

Turning points:

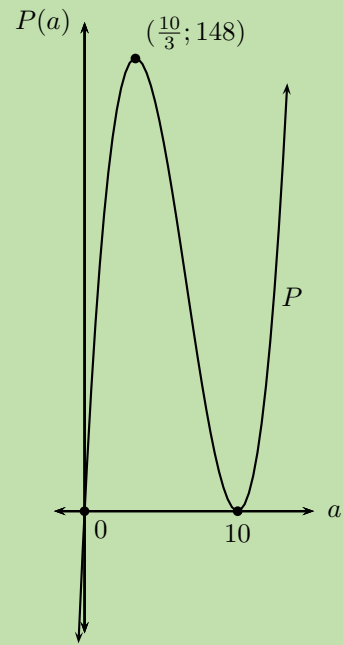
$$\begin{aligned} P'(a) &= 0 \\ \therefore a &= \frac{10}{3} \text{ or } a = 10 \end{aligned}$$

Maximum and minimum function values:

$$\text{Substitute } \left(\frac{10}{3}; \frac{20}{3}\right): \quad P = ab^2$$
$$= \left(\frac{10}{3}\right)\left(\frac{20}{3}\right)^2$$
$$= \frac{4000}{27}$$
$$\approx 148$$

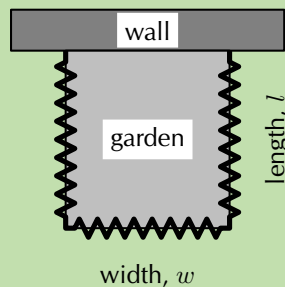
(A maximum turning point)

$$\text{Substitute } (0; 10): \quad P = ab^2$$
$$= (10)(0)^2$$
$$= 0$$

(A minimum turning point)

Note: the above diagram is not drawn to scale.

Worked example 22: Optimisation problems

QUESTION

Michael wants to start a vegetable garden, which he decides to fence off in the shape of a rectangle from the rest of the garden. Michael has only 160 m of fencing, so he decides to use a wall as one border of the vegetable garden. Calculate the width and length of the garden that corresponds to the largest possible area that Michael can fence off.

SOLUTION

Step 1: Examine the problem and formulate the equations that are required

The important pieces of information given are related to the area and modified perimeter of the garden. We know that the area of the garden is given by the formula:

$$\text{Area } = w \times l$$

The fencing is only required for 3 sides and the three sides must add up to 160 m.

$$160 = w + l + l$$

Rearrange the formula to make w the subject of the formula:

$$w = 160 - 2l$$

Substitute the expression for w into the formula for the area of the garden. Notice that this formula now contains only one unknown variable.

$$\begin{aligned} \text{Area } &= l(160 - 2l) \\ &= 160l - 2l^2 \end{aligned}$$

Step 2: Differentiate with respect to l

We are interested in maximising the area of the garden, so we differentiate to get the following:

$$\frac{dA}{dl} = A' = 160 - 4l$$

Step 3: Calculate the stationary point

To find the stationary point, we set $A'(l) = 0$ and solve for the value(s) of l that maximises the area:

$$\begin{aligned} A'(l) &= 160 - 4l \\ 0 &= 160 - 4l \\ 4l &= 160 \\ \therefore l &= 40 \end{aligned}$$

Therefore, the length of the garden is 40 m.

Substitute to solve for the width:

$$\begin{aligned} w &= 160 - 2l \\ &= 160 - 2(40) \\ &= 160 - 80 \\ &= 80 \end{aligned}$$

Therefore, the width of the garden is 80 m.

Step 4: Determine the second derivative $A''(l)$

We can check that this gives a maximum area by showing that $A''(l) < 0$:

$$A''(l) = -4$$

Step 5: Write the final answer

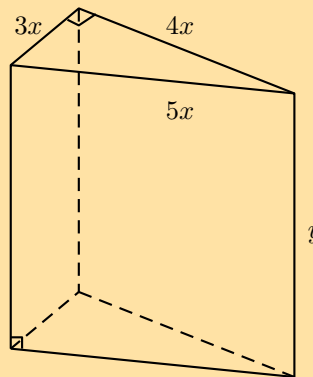
A width of 80 m and a length of 40 m will give the maximum area for the garden.

Important note:

The quantity that is to be minimised or maximised must be expressed in terms of only one variable. To find the optimised solution we need to determine the derivative and we only know how to differentiate with respect to one variable (more complex rules for differentiation are studied at university level).

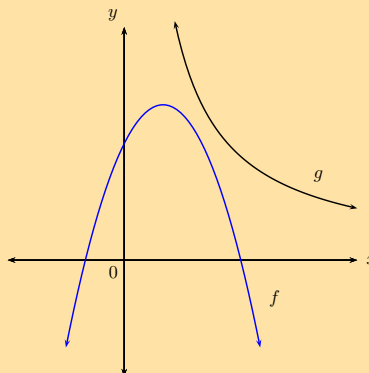
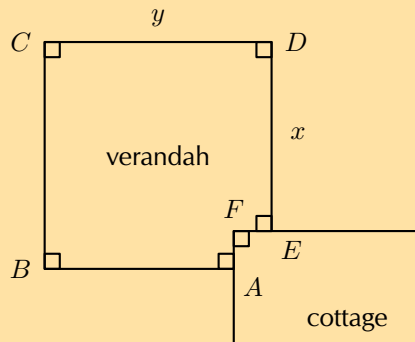
Exercise 6 – 11: Solving optimisation problems

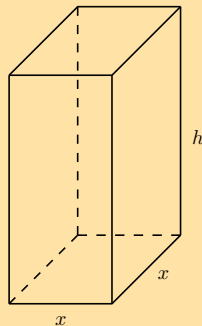
1. The sum of two positive numbers is 20. One of the numbers is multiplied by the square of the other. Find the numbers that make this product a maximum.

2. A wooden block is made as shown in the diagram. The ends are right-angled triangles having sides $3x$, $4x$ and $5x$. The length of the block is y. The total surface area of the block is 3600 cm^2.

   a) Show that $y = \frac{300-x^2}{x}$.
   b) Find the value of x for which the block will have a maximum volume.
   (Volume = area of base × height)

3. Determine the shortest vertical distance between the curves of f and g if it is given that:

$$f(x) = -x^2 + 2x + 3$$

$$\text{and } g(x) = \frac{8}{x}, \quad x > 0$$

4. The diagram shows the plan for a verandah which is to be built on the corner of a cottage. A railing $ABCDE$ is to be constructed around the four edges of the verandah.

If $AB = DE = x$ and $BC = CD = y$, and the length of the railing must be 30 m, find the values of x and y for which the verandah will have a maximum area.

5. A rectangular juice container, made from cardboard, has a square base and holds 750 cm^3 of juice. The container has a specially designed top that folds to close the container. The cardboard needed to fold the top of the container is twice the cardboard needed for the base, which only needs a single layer of cardboard.

   a) If the length of the sides of the base is x cm, show that the total area of the cardboard needed for one container is given by:

$$A(\text{in square centimetres}) = \frac{3000}{x} + 3x^2$$

   b) Determine the dimensions of the container so that the area of the cardboard used is minimised.

6. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 28YN 2a. 28YP 2b. 28YQ 3. 28YR 4. 28YS 5a. 28YT
5b. 28YV

Rates of change EMCHK

It is very useful to determine how fast (the rate at which) things are changing. Mathematically we can represent change in different ways. For example we can use algebraic formulae or graphs.

Graphs give a visual representation of the rate at which the function values change as the independent (input) variable changes. This rate of change is described by the gradient of the graph and can therefore be determined by calculating the derivative.

We have learnt how to determine the average gradient of a curve and how to determine the gradient of a curve at a given point. These concepts are also referred to as the average rate of change and the instantaneous rate of change.

$$\begin{aligned} \text{Average rate of change } &= \frac{f(x+h) - f(x)}{(x+h) - x} \\ \text{Instantaneous rate of change } &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \end{aligned}$$

When we mention rate of change, the instantaneous rate of change (the derivative) is implied. When average rate of change is required, it will be specifically referred to as average rate of change.

Velocity is one of the most common forms of rate of change:

$$\begin{aligned} \text{Average velocity } &= \text{Average rate of change} \\ \text{Instantaneous velocity } &= \text{Instantaneous rate of change} \\ &= \text{Derivative} \end{aligned}$$

Velocity refers to the change in distance (s) for a corresponding change in time (t).

$$v(t) = \frac{ds}{dt} = s'(t)$$

Acceleration is the change in velocity for a corresponding change in time. Therefore, acceleration is the derivative of velocity

$$a(t) = v'(t)$$

This implies that acceleration is the second derivative of the distance.

$$a(t) = s''(t)$$

Worked example 23: Rate of change

QUESTION

The height (in metres) of a golf ball t seconds after it has been hit into the air, is given by $H(t) = 20t - 5t^2$. Determine the following:

1. The average vertical velocity of the ball during the first two seconds.
2. The vertical velocity of the ball after $1{,}5$ s.
3. The time at which the vertical velocity is zero.
4. The vertical velocity with which the ball hits the ground.
5. The acceleration of the ball.

SOLUTION

Step 1: Determine the average vertical velocity during the first two seconds

$$\begin{aligned} v_{\text{ave}} &= \frac{H(2) - H(0)}{2 - 0} \\ &= \frac{\left[20(2) - 5(2)^2\right] - \left[20(0) - 5(0)^2\right]}{2} \\ &= \frac{40 - 20}{2} \\ &= 10 \text{ m.s}^{-1} \end{aligned}$$

Step 2: Calculate the instantaneous vertical velocity

$$\begin{aligned} v(t) &= H'(t) \\ &= \frac{dH}{dt} \\ &= 20 - 10t \end{aligned}$$

Velocity after $1{,}5$ s:

$$\begin{aligned} v(1{,}5) &= 20 - 10(1{,}5) \\ &= 5 \text{ m.s}^{-1} \end{aligned}$$

Step 3: Determine the time at which the vertical velocity is zero

$$\begin{aligned} v(t) &= 0 \\ 20 - 10t &= 0 \\ 10t &= 20 \\ t &= 2 \end{aligned}$$

Therefore, the velocity is zero after 2 s

Step 4: Find the vertical velocity with which the ball hits the ground

The ball hits the ground when $H(t) = 0$

$$\begin{aligned} 20t - 5t^2 &= 0 \\ 5t(4 - t) &= 0 \\ t &= 0 \text{ or } t = 4 \end{aligned}$$

The ball hits the ground after 4 s. The velocity after 4 s will be:

$$\begin{aligned} v(4) &= H'(4) \\ &= 20 - 10(4) \\ &= -20 \text{ m.s}^{-1} \end{aligned}$$

The ball hits the ground at a speed of 20 m.s^{-1}. Notice that the sign of the velocity is negative which means that the ball is moving downward (a positive velocity is used for upwards motion).

Step 5: Acceleration

$$\begin{aligned} a = v'(t) &= H''(t) \\ &= -10 \end{aligned}$$

$$\therefore a = -10 \text{ m.s}^{-2}$$

Just because gravity is constant does not mean we should necessarily think of acceleration as a constant. We should still consider it a function.

Exercise 6 – 12: Rates of change

1. A pump is connected to a water reservoir. The volume of the water is controlled by the pump and is given by the formula:

$$\begin{aligned} V(d) &= 64 + 44d - 3d^2 \\ \text{where } V &= \text{ volume in kilolitres} \\ d &= \text{ days} \end{aligned}$$

   a) Determine the rate of change of the volume of the reservoir with respect to time after 8 days.
   b) Is the volume of the water increasing or decreasing at the end of 8 days. Explain your answer.
   c) After how many days will the reservoir be empty?
   d) When will the amount of water be at a maximum?
   e) Calculate the maximum volume.
   f) Draw a graph of $V(d)$.

2. A soccer ball is kicked vertically into the air and its motion is represented by the equation:

$$D(t) = 1 + 18t - 3t^2$$

where $D =$ distance above the ground (in metres)

$t =$ time elapsed (in seconds)

   a) Determine the initial height of the ball at the moment it is being kicked.
   b) Find the initial velocity of the ball.
   c) Determine the velocity of the ball after $1{,}5$ s.
   d) Calculate the maximum height of the ball.
   e) Determine the acceleration of the ball after 1 second and explain the meaning of the answer.
   f) Calculate the average velocity of the ball during the third second.
   g) Determine the velocity of the ball after 3 seconds and interpret the answer.
   h) How long will it take for the ball to hit the ground?
   i) Determine the velocity of the ball when it hits the ground.

3. If the displacement s (in metres) of a particle at time t (in seconds) is governed by the equation $s = \frac{1}{2}t^3 - 2t$, find its acceleration after 2 seconds.
4. During an experiment the temperature T (in degrees Celsius) varies with time t (in hours) according to the formula: $T(t) = 30 + 4t - \frac{1}{2}t^2, \quad t \in [1; 10]$.

   a) Determine an expression for the rate of change of temperature with time.
   b) During which time interval was the temperature dropping?

5. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 28YW 2a. 28YX 2b. 28YY 2c. 28YZ 2d. 28Z2 2e. 28Z3
2f. 28Z4 2g. 28Z5 2h. 28Z6 2i. 28Z7 3. 28Z8 4a. 28Z9
4b. 28ZB

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6.8 Summary EMCHM

- The limit of a function exists and is equal to L if the values of $f(x)$ get closer to L from both sides as x gets closer to a.

$$\lim_{x \to a} f(x) = L$$

- **Average gradient or average rate of change:**

$$\text{Average gradient } = \frac{f(x+h) - f(x)}{h}$$

- **Gradient at a point or instantaneous rate of change:**

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- **Notation**

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}[f(x)] = Df(x) = D_x y$$

- **Differentiating from first principles:**

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- **Rules for differentiation:**
  - General rule for differentiation:

  $$\frac{d}{dx}[x^n] = nx^{n-1}, \text{ where } n \in \mathbb{R} \text{ and } n \neq 0.$$

  - The derivative of a constant is equal to zero.

  $$\frac{d}{dx}[k] = 0$$

  - The derivative of a constant multiplied by a function is equal to the constant multiplied by the derivative of the function.

  $$\frac{d}{dx}[k \,.\, f(x)] = k\frac{d}{dx}[f(x)]$$

  - The derivative of a sum is equal to the sum of the derivatives.

  $$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$$

  - The derivative of a difference is equal to the difference of the derivatives.

  $$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)]$$

- **Second derivative:**

$$f''(x) = \frac{d}{dx}[f'(x)]$$

- **Sketching graphs:**
  The gradient of the curve and the tangent to the curve at stationary points is zero.
  Finding the stationary points: let $f'(x) = 0$ and solve for x.
  A stationary point can either be a local maximum, a local minimum or a point of inflection.
- **Optimisation problems:**
  Use the given information to formulate an expression that contains only one variable.
  Differentiate the expression, let the derivative equal zero and solve the equation.

Exercise 6 – 13: End of chapter exercises

1. Determine $f'(x)$ from first principles if $f(x) = 2x - x^2$.
2. Given $f(x) = \frac{1}{x} + 3$, find $f'(x)$ using the definition of the derivative.
3. Calculate: $\lim_{x \to 1} \dfrac{1 - x^3}{1 - x}$
4. Determine $\frac{dy}{dx}$ if:
   a) $y = (x + 2)(7 - 5x)$
   b) $y = \frac{8x^3+1}{2x+1}$
   c) $y = (2x)^2 - \frac{1}{3x}$
   d) $y = \frac{2\sqrt{x}-5}{\sqrt{x}}$
5. Given: $f(x) = 2x^2 - x$
   a) Use the definition of the derivative to calculate $f'(x)$.
   b) Hence, calculate the coordinates of the point at which the gradient of the tangent to the graph of f is 7.
6. If $g(x) = \left(x^{-2} + x^2\right)^2$, calculate $g'(2)$.
7. Given: $f(x) = 2x - 3$
   a) Find $f^{-1}(x)$.
   b) Solve $f^{-1}(x) = 3f'(x)$.
8. Find the derivative for each of the following:
   a) $p(t) = \frac{\sqrt[5]{t^3}}{3} + 10$
   b) $k(n) = \frac{\left(2n^2-5\right)(3n+2)}{n^2}$
9. If $xy - 5 = \sqrt{x^3}$, determine $\frac{dy}{dx}$.
10. Given: $y = x^3$
    a) Determine $\frac{dy}{dx}$.
    b) Find $\frac{dx}{dy}$.
    c) Show that $\frac{dy}{dx} \times \frac{dx}{dy} = 1$.
11. Given: $f(x) = x^3 - 3x^2 + 4$
    a) Calculate $f(-1)$.
    b) Hence, solve $f(x) = 0$.
    c) Determine $f'(x)$.
    d) Sketch the graph of f, showing the coordinates of the turning points and the intercepts on both axes.
    e) Determine the coordinates of the points on the graph of f where the gradient is 9.
    f) Draw the graph of $f'(x)$ on the same system of axes.
    g) Determine $f''(x)$ and use this to make conclusions about the concavity of f.
12. Given $f(x) = 2x^3 - 5x^2 - 4x + 3$.
    a) If $f(-1) = 0$, determine the x-intercepts of f.
    b) Determine the coordinates of the turning points of f.

c) Draw a sketch graph of f. Clearly indicate the coordinates of the turning points and the intercepts with the axes.

d) For which value(s) of k will the equation $f(x) = k$ have three real roots of which two are equal?

e) Determine the equation of the tangent to the graph of $f(x) = 2x^3 - 5x^2 - 4x + 3$ at the point where $x = 1$.

13. Given the function $f(x) = x^3 + bx^2 + cx + d$ with y-intercept $(0; 26)$, x-intercept $(-2; 0)$ and a point of inflection at $x = -3$.

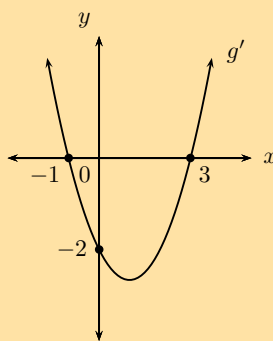
a) Show by calculation that $b = 9$, $c = 27$ and $d = 26$.

b) Find the y-coordinate of the point of inflection.

c) Draw the graph of f.

d) Discuss the gradient of f.

e) Discuss the concavity of f.

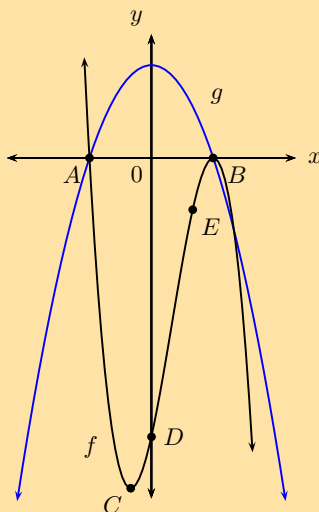
14. The sketch shows the graph of $g'(x)$.

a) Identify the stationary points of the cubic function, $g(x)$.

b) What is the gradient of function g where $x = 0$.

c) If it is further given that $g(x)$ has only two real roots, draw a rough sketch of $g(x)$. Intercept values do not need to be shown.

15. Given that $h(x)$ is a linear function with $h(2) = 11$ and $h'(2) = -1$, find the equation of $h(x)$.

16. The graphs of f and g and the following points are given below:

$$A(-3; 0) \quad B(3; 0) \quad C(-1; -32) \quad D(0; -27) \quad E(2; y)$$

a) Use the graphs and determine the values of x for which:

i. $f(x)$ is a decreasing function.

ii. $f(x)\ .\ g(x) \geq 0$.

iii. $f'(x)$ and $g(x)$ are both negative.

b) Given $f(x) = -x^3 + 3x^2 + 9x - 27$, determine the equation of the tangent to f at the point $E(2; y)$.

c) Find the coordinates of the point(s) where the tangent in the question above meets the graph of f again.

d) Without any calculations, give the x-intercepts of the graph of $f'(x)$. Explain reasoning.

17. a) Sketch the graph of $f(x) = x^3 - 9x^2 + 24x - 20$, show all intercepts with the axes and turning points.

b) Find the equation of the tangent to $f(x)$ at $x = 4$.

c) Determine the point of inflection and discuss the concavity of f.

18. Determine the minimum value of the sum of a positive number and its reciprocal.

19. t minutes after a kettle starts to boil, the height of the water in the kettle is given by $d = 86 - \frac{1}{8}t - \frac{1}{4}t^3$, where d is measured in millimetres.

a) Calculate the height of the water level in the kettle just before it starts to boil.

b) As the water boils, the water level in the kettle decreases. Determine the rate at which the water level is decreasing when $t = 2$ minutes.

c) How many minutes after the kettle starts to boil will the water level be decreasing at a rate of $12\frac{1}{8}$ mm per minute?

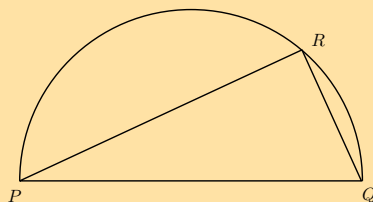
20. The displacement of a moving object is represented by the equation:

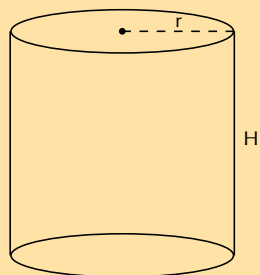
$$D(t) = \frac{4}{3}t^3 - 3t$$

where $D =$ distance travelled in metres

$t =$ time in seconds

Calculate the acceleration of the object after 3 seconds.

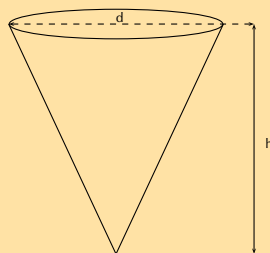
21. In the figure PQ is the diameter of the semi-circle PRQ. The sum of the lengths of PR and QR is 10 units. Calculate the perimeter of $\triangle PQR$ when $\triangle PQR$ covers the maximum area in the semi-circle. Leave the answer in simplified surd form.

22. The capacity of a cylindrical water tank is 1000 litres. Let the height be H and the radius be r. The material used for the bottom of the tank is twice as thick and also twice as expensive as the material used for the curved part of the tank and the top of the tank.

    Remember: $1000\ \ell = 1$ m^3

    a) Express H in terms of r.
    b) Show that the cost of the material for the tank can be expressed as:

    $$C = 3\pi r^2 + \frac{2}{r}$$

    c) Determine the diameter of the tank that gives the minimum cost of the materials.

    [IEB, 2006]

23. The diameter of an icecream cone is d and the vertical height is h. The sum of the diameter and the height of the cone is 10 cm.

    a) Determine the volume of the cone in terms of h and d.

       (Volume of a cone: $V = \frac{1}{3}\pi r^3 h$)
    b) Determine the radius and height of the cone for the volume to be a maximum.
    c) Calculate the maximum volume of the cone.

24. A water reservoir has both an inlet and an outlet pipe to regulate the depth of the water in the reservoir. The depth is given by the function:

    $$D(h) = 3 + \frac{1}{2}h - \frac{1}{4}h^3$$

    where $D =$ depth in metres

    $h =$ hours after 06h00

    a) Determine the rate at which the depth of the water is changing at 10h00.
    b) Is the depth of the water increasing or decreasing?
    c) At what time will the inflow of water be the same as the outflow?

    [IEB, 2006]

25. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 28ZC	2. 28ZD	3. 28ZF	4a. 28ZG	4b. 28ZH	4c. 28ZJ
4d. 28ZK	5a. 28ZM	5b. 28ZN	6. 28ZP	7a. 28ZQ	7b. 28ZR
8a. 28ZS	8b. 28ZT	9. 28ZV	10a. 28ZW	10b. 28ZX	10c. 28ZY
11a. 28ZZ	11b. 2922	11c. 2923	11d. 2924	11e. 2925	11f. 2926
11g. 2927	12a. 2928	12b. 2929	12c. 292B	12d. 292C	12e. 292D
13a. 292F	13b. 292G	13c. 292H	13d. 292J	13e. 292K	14a. 292M
14b. 292N	14c. 292P	15. 292Q	16a. 292R	16b. 292S	16c. 292T
16d. 292V	17. 292W	18. 292X	19a. 292Y	19b. 292Z	19c. 2932
20. 2933	21. 2934	22a. 2935	22b. 2936	22c. 2937	23a. 2938
23b. 2939	23c. 293B	24a. 293C	24b. 293D	24c. 293F	

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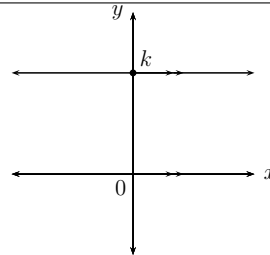
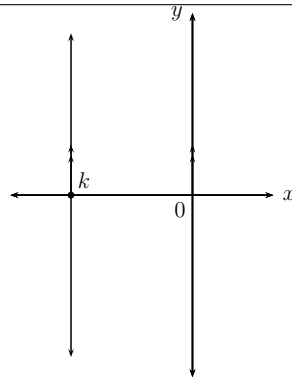
CHAPTER

Analytical geometry

7.1 Revision EMCHN

Straight line equations EMCHP

Theorem of Pythagoras:	$AB^2 = AC^2 + BC^2$
Distance formula:	$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
Gradient:	$m_{AB} = \frac{y_2 - y_1}{x_2 - x_1}$ or $m_{AB} = \frac{y_1 - y_2}{x_1 - x_2}$
Mid-point of a line segment:	$M(x; y) = \left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right)$
Points on a straight line:	$m_{AB} = m_{AM} = m_{MB}$

Two-point form:	$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$	
Gradient-point form:	$y - y_1 = m(x - x_1)$	
Gradient-intercept form:	$y = mx + c$	

Horizontal lines:	$y = k$	
Vertical lines:	$x = k$	

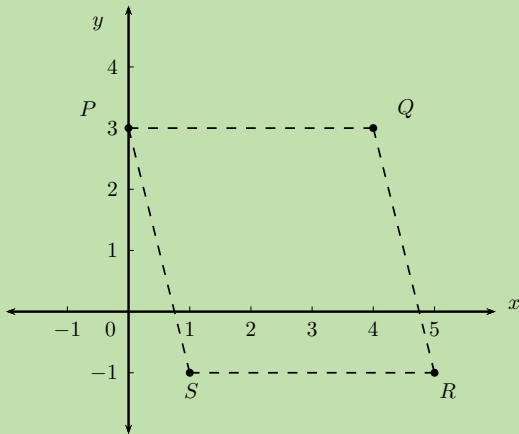
Worked example 1: Revision

QUESTION

Given quadrilateral $PQRS$ with vertices $P(0;3), Q(4;3), R(5;-1)$ and $S(1;-1)$.

1. Determine the equation of the lines PS and QR.
2. Show that $PS \parallel QR$.
3. Calculate the lengths of PS and QR.
4. Determine the equation of the diagonal QS.
5. What type of quadrilateral is $PQRS$?

SOLUTION

Step 1: Draw a sketch

Step 2: Use the given information to determine the equation of lines PS and QR

$$\text{Gradient: } m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\text{Two-point form: } \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\text{Gradient-intercept form: } y = mx + c$$

Determine the equation of the line PS using the two point form of the straight line equation:

$$x_1 = 0; \qquad y_1 = 3; \qquad x_2 = 1; \qquad y_2 = -1$$

$$\begin{aligned}
\frac{y - y_1}{x - x_1} &= \frac{y_2 - y_1}{x_2 - x_1} \\
\frac{y - 3}{x - 0} &= \frac{-1 - 3}{1 - 0} \\
\frac{y - 3}{x} &= -4 \\
y - 3 &= -4x \\
\therefore y &= -4x + 3
\end{aligned}$$

Determine the equation of the line QR using the gradient-intercept form of the straight line equation:

$$\begin{aligned}
m_{QR} &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{-1 - 3}{5 - 4} \\
&= \frac{-4}{1} \\
&= -4 \\
y &= mx + c \\
y &= -4x + c \\
\text{Substitute } (4;3) \quad 3 &= -4(4) + c \\
\therefore c &= 19 \\
y &= -4x + 19
\end{aligned}$$

There is often more than one method for determining the equation of a line. The different forms of the straight line equation are used, depending on the information provided in the problem.

Step 3: Show that line PS and line QR have equal gradients

$$\begin{aligned}
y &= -4x + 3 \\
\therefore m_{PS} &= -4 \\
\text{And } y &= -4x + 19 \\
\therefore m_{QR} &= -4 \\
\therefore m_{PS} &= m_{QR} \\
\therefore PS &\parallel QR
\end{aligned}$$

Step 4: Use the distance formula to determine the lengths of PS and QR

$$\begin{aligned} PS &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(1-0)^2 + (-1-3)^2} \\ &= \sqrt{1 + (-4)^2} \\ &= \sqrt{17} \text{ units} \end{aligned}$$

$$\begin{aligned} QR &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(5-4)^2 + (-1-3)^2} \\ &= \sqrt{1 + (-4)^2} \\ &= \sqrt{17} \text{ units} \end{aligned}$$

Step 5: Determine the equation of the diagonal QS

Determine the gradient of the line:

$$\begin{aligned} m_{QS} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-1-3}{1-4} \\ &= \frac{-4}{-3} \\ &= \frac{4}{3} \end{aligned}$$

Use gradient and the point $Q(4;3)$ to determine the equation of the line QS:

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - y_1 &= \frac{4}{3}(x - x_1) \\ \text{Substitute } (4;3) \quad y - 3 &= \frac{4}{3}(x - 4) \\ y - 3 &= \frac{4}{3}x - \frac{16}{3} \\ y &= \frac{4}{3}x - \frac{16}{3} + 3 \\ \therefore y &= \frac{4}{3}x - \frac{7}{3} \end{aligned}$$

Step 6: Examine the properties of quadrilateral $PQRS$

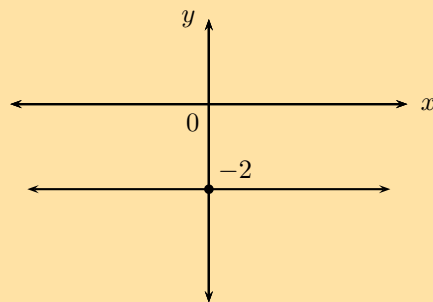
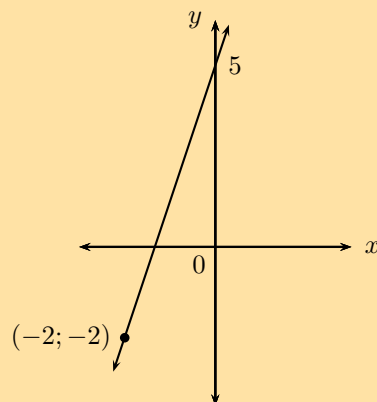
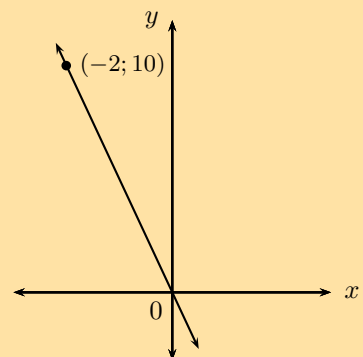
We have shown that $PS \parallel QR$ and $PS = QR$, therefore quadrilateral $PQRS$ is a parallelogram (one pair of opposite sides equal and parallel).

Exercise 7 – 1: Revision

1. Determine the following for the line segment between the given points:

   - length
   - mid-point
   - gradient
   - equation

   a) $(-2;-4)$ and $(3;11)$
   b) $(-5;-3)$ and $(10;6)$
   c) $(h;-h-k)$ and $(2k;h-5k)$
   d) $(2;9)$ and $(0;-1)$

2. The line joining $A(x;y)$ and $B(-3;6)$ has the mid-point $M(2;3)$. Determine the values of x and y.
3. Given $F(2;11), G(-4;r)$ and length $FG = 6\sqrt{5}$ units, determine the value(s) of r.
4. Determine the equation of the straight line:
   a) passing through the point $(\frac{1}{2};4)$ and $(1;5)$.
   b) passing through the points $(2;-3)$ and $(-1;0)$.
   c) passing through the point $(9;1)$ and with $m = \frac{1}{3}$.
   d) parallel to the x-axis and passing through the point $(0;-4)$.
   e) passing through the point $(\frac{1}{2};-1)$ and with $m = -4$.
   f) perpendicular to the x-axis and passing through the point $(5;0)$.
   g) with undefined gradient and passing through the point $(\frac{3}{4};0)$.
   h) with $m = 2p$ and passing through the point $(3;6p+3)$.
   i) which cuts the y-axis at $y = -\frac{3}{5}$ and with $m = 4$.
   j)

   k)

   l)

5. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 293J	1b. 293K	1c. 293M	1d. 293N	2. 293P	3. 293Q
4a. 293R	4b. 293S	4c. 293T	4d. 293V	4e. 293W	4f. 293X
4g. 293Y	4h. 293Z	4i. 2942	4j. 2943	4k. 2944	4l. 2945

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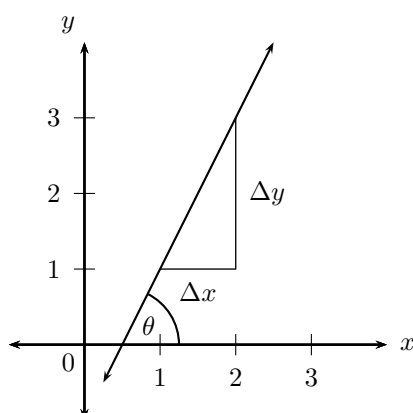
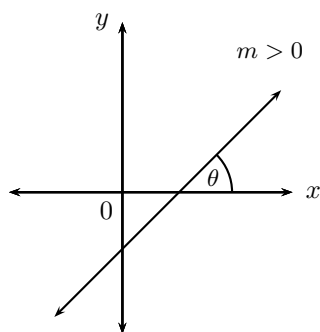
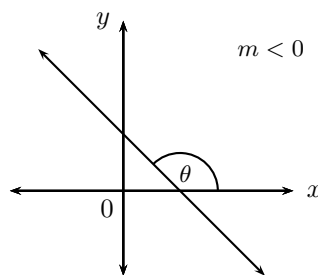

The diagram shows a straight line which forms an acute angle θ with the positive x-axis. This is called the **angle of inclination** of a straight line.

The gradient of a straight line is equal to the tangent of the angle formed between the line and the positive direction of the x-axis.

$$m = \tan\theta \qquad \text{for } 0° \leq \theta < 180°$$

θ is an acute angle

θ is an obtuse angle

Lines with positive gradients

A line with a positive gradient ($m > 0$) has an acute angle of inclination ($0° < \theta < 90°$).

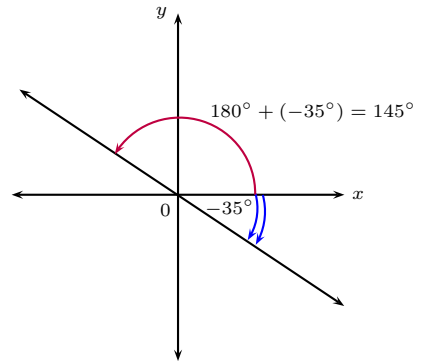
For example, we can determine the angle of inclination of a line with $m = 1{,}2$:

$$\begin{aligned} \tan\theta &= m \\ &= 1{,}2 \\ \therefore \theta &= \tan^{-1}(1{,}2) \\ &= 50{,}2° \end{aligned}$$

Lines with negative gradients

If we are calculating the angle of inclination of a line with a negative gradient ($m < 0$), then we add $180°$ to change the negative angle to an obtuse angle ($90° < \theta < 180°$).

For example, we can determine the angle of inclination for a line with $m = -0{,}7$:

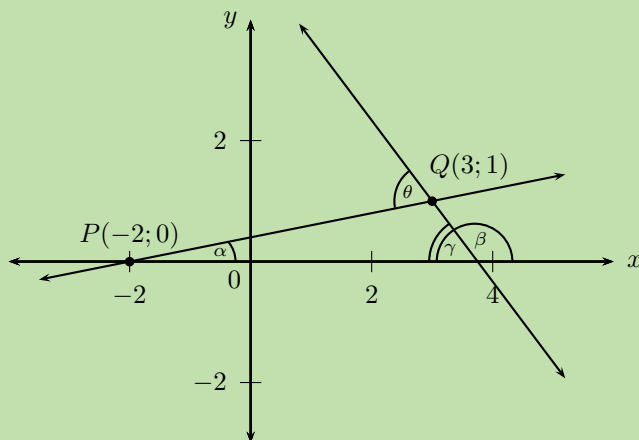
$$
\begin{aligned}
\tan\theta &= m \\
&= -0{,}7 \\
\therefore \theta &= \tan^{-1}(-0{,}7) \\
&= -35{,}0° \\
\text{Obtuse angle: } \theta &= -35{,}0° + 180° \\
&= 145°
\end{aligned}
$$

Worked example 2: Inclination of a straight line

QUESTION

Determine the acute angle (correct to 1 decimal place) between the line passing through the points $P(-2;0)$ and $Q(3;1)$ and the straight line $y = -\frac{4}{3}x + 5$.

SOLUTION

Step 1: Draw a sketch

Draw the line through points $P(-2;0)$ and $Q(3;1)$ and the line $y = -\frac{4}{3}x + 5$ on a suitable system of axes. Label α and β, the angles of inclination of the two lines. Label θ, the acute angle between the two straight lines.

Notice that α and θ are acute angles and β is an obtuse angle.

$$\gamma = 180° - \beta \qquad (\angle \text{ on str. line})$$

$$\text{and } \theta = \alpha + \gamma \qquad (\text{ext. } \angle \text{ of } \triangle = \text{ sum int. opp})$$

$$\therefore \theta = \alpha + (180° - \beta)$$

$$= 180° + \alpha - \beta$$

Step 2: Use the gradient to determine the angle of inclination β

From the equation $y = -\frac{4}{3}x + 5$ we see that $m < 0$, therefore β is an obtuse angle.

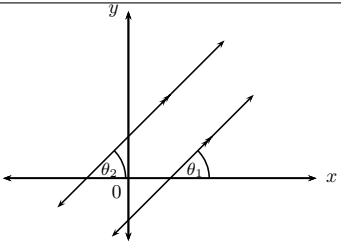
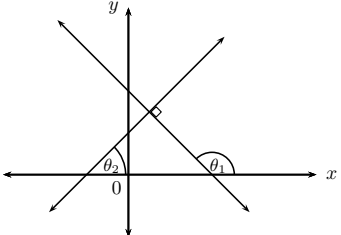
$$\begin{aligned} m &= -\frac{4}{3} \\ \tan\beta &= -\frac{4}{3} \\ \therefore \beta &= \tan^{-1}\left(-\frac{4}{3}\right) \\ &= -53{,}1° \\ \beta &= -53{,}1° + 180° \\ &= 126{,}9° \end{aligned}$$

Step 3: Determine the gradient and angle of inclination of the line through P and Q

Determine the gradient

$$\begin{aligned} m &= \frac{y_P - y_Q}{x_P - x_Q} \\ &= \frac{-1}{-5} \\ &= \frac{1}{5} \end{aligned}$$

Determine the angle of inclination

$$\begin{aligned} \tan\alpha &= m \\ &= \frac{1}{5} \\ \therefore \alpha &= \tan^{-1}\left(\frac{1}{5}\right) \\ &= 11{,}3° \end{aligned}$$

Step 4: Write the final answer

$$\begin{aligned} \theta &= 180° + \alpha - \beta \\ &= 180° + 11{,}3° - 126{,}9° \\ &= 64{,}4° \end{aligned}$$

The acute angle between the two straight lines is $64{,}4°$.

Parallel lines		$m_1 = m_2$	$\theta_1 = \theta_2$
Perpendicular lines		$m_1 \times m_2 = -1$	$\theta_1 = 90° + \theta_2$

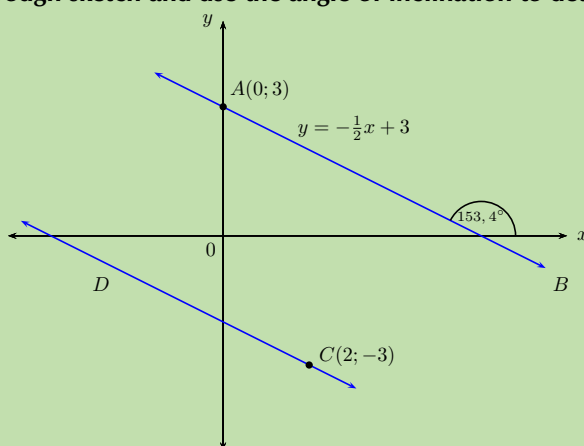
Worked example 3: Parallel lines

QUESTION

Line AB passes through the point $A(0;3)$ and has an angle of inclination of $153{,}4°$.

1. Determine the equation of line CD which passes through the point $C(2;-3)$ and is parallel to AB.
2. Determine the equation of line EF, which passes through the origin and is perpendicular to both AB and CD.
3. Sketch lines AB, CD and EF on the same system of axes.
4. Use two different methods to determine the angle of inclination of EF.

SOLUTION

Step 1: Draw a rough sketch and use the angle of inclination to determine the equation of CD

$$
\begin{aligned}
m_{AB} &= \tan\theta \\
&= \tan 153{,}4° \\
&= -0{,}5
\end{aligned}
$$

Since we are given $AB \parallel CD$,

$$m_{CD} = m_{AB} = -0{,}5 = -\frac{1}{2}$$
$$y - y_1 = m(x - x_1)$$
$$y - y_1 = -\frac{1}{2}(x - x_1)$$

Substitute the given point $(2; -3)$:

$$y - (-3) = -\frac{1}{2}(x - 2)$$
$$y + 3 = -\frac{1}{2}x + 1$$
$$y = -\frac{1}{2}x - 2$$

Step 2: Determine the equation of EF

EF is perpendicular to AB, therefore the product of their gradients is equal to -1:

$$m_{AB} \times m_{EF} = -1$$
$$-\frac{1}{2} \times m_{EF} = -1$$
$$\therefore m_{EF} = 2$$

We know line EF passes through $(0; 0)$, therefore the equation of the line is:

$$y = 2x$$

Step 3: Determine the angle of inclination of EF

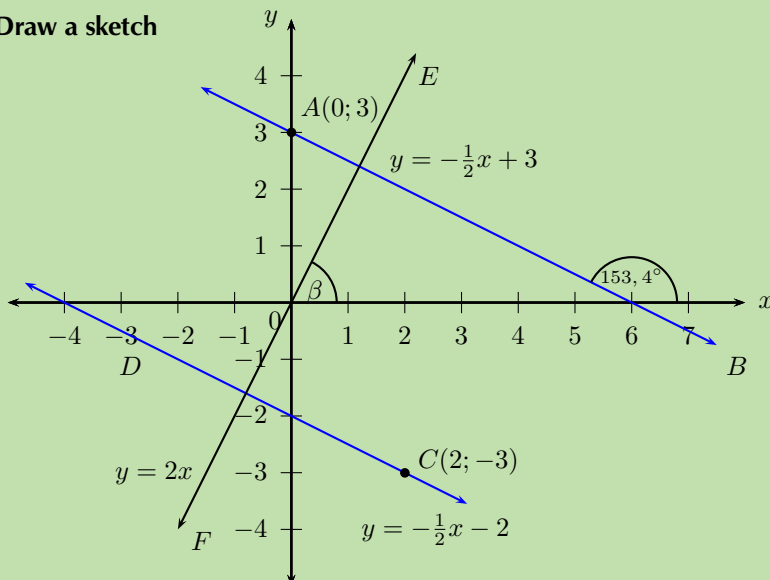
Let the angle of inclination of EF be β.

Method 1:

$$\beta = 153{,}4^\circ - 90^\circ$$
$$= 63{,}4^\circ$$

Method 2:

$$m = 2$$
$$\tan\beta = 2$$
$$\therefore \beta = 63{,}4^\circ$$

Step 4: Draw a sketch

Exercise 7 – 2: Inclination of a straight line

1. Determine the angle of inclination (correct to 1 decimal place) for each of the following:

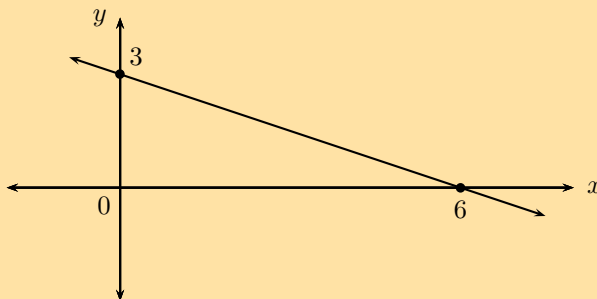
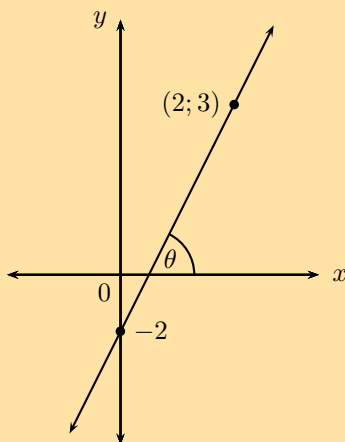
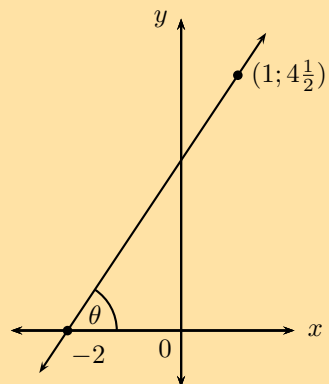
   a) a line with $m = \frac{3}{4}$

   b) $6 + x = 2y$

   c) the line passes through the points $(-4; 0)$ and $(2; 6)$

   d) $y = 4$

   e) a line with a gradient of $1{,}733$

   f)

   g)

   h)

2. Find the angle between the line $2y = 5x$ and the line passing through points $T(2; 1\frac{1}{3})$ and V$(-3; 3)$.
3. Determine the equation of the straight line that passes through the point $(1; 2)$ and is parallel to the line $y + 3x = 1$.
4. Determine the equation of the straight line that passes through the point $(-4; -4)$ and is parallel to the line with angle of inclination $\theta = 56{,}31°$.
5. Determine the equation of the straight line that passes through the point $(1; -6)$ and is perpendicular to the line $5y = x$.
6. Determine the equation of the straight line that passes through the point $(3; -1)$ and is perpendicular to the line with angle of inclination $\theta = 135°$.
7. $A(2; 3)$, $B(-4; 0)$ and $C(5; -3)$ are the vertices of $\triangle ABC$ in the Cartesian plane. AC intersects the x-axis at D. Draw a sketch and determine the following:

   a) the equation of line AC

   b) the coordinates of point D

c) the angle of inclination of AC

d) the gradient of line AB

e) $B\hat{A}C$

f) the equation of the line perpendicular to AB and passing through the origin

g) the mid-point M of BC

h) the equation of the line parallel to AC and passing through point M

8. Points $F(-3;5)$, $G(-7;-4)$ and $H(2;0)$ are given.

   a) Plot the points on the Cartesian plane.

   b) Determine the coordinates of I if $FGHI$ is a parallelogram.

   c) Prove that $FGHI$ is a rhombus.

9. Given points $S(2;5)$, $T(-3;-4)$ and $V(4;-2)$.

   a) Show that the equation of the line ST is $5y = 9x + 7$.

   b) Determine the size of $T\hat{S}V$.

10. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 2946	1b. 2947	1c. 2948	1d. 2949	1e. 294B	1f. 294C
1g. 294D	1h. 294F	2. 294G	3. 294H	4. 294J	5. 294K
6. 294M	7. 294N	8. 294P	9. 294Q		

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7.2 Equation of a circle EMCHS

Equation of a circle with centre at the origin EMCHT

Investigation:

1. Draw a system of axes with a scale of 1 cm = 1 unit on the x-axis and on the y-axis.
2. Draw the lines $y = x$ and $y = -x$.
3. Plot the following points:

   a) $O(0;0)$

   b) $D(2;0)$

   c) $E(\sqrt{2};\sqrt{2})$

   d) $F(0;2)$

   e) $G(-\sqrt{2};\sqrt{2})$

   f) $H(-2;0)$

   g) $I(-\sqrt{2};-\sqrt{2})$

   h) $J(0;-2)$

   i) $K(\sqrt{2};-\sqrt{2})$

   What object do the points form?

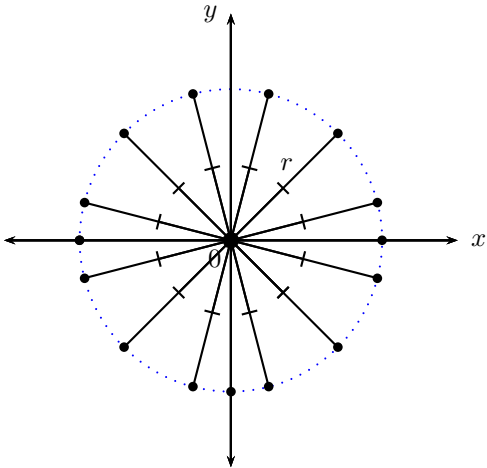
4. Measure the following distances and complete the table below:

Line segment	Distance (cm)
DO	
EO	
FO	
GO	
HO	
IO	
JO	
KO	

5. Use the distance formula to check the results of the above table:

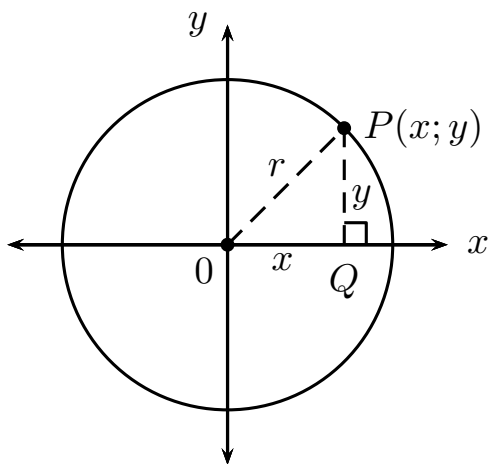
$$\text{Distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Line segment	Distance
DO	
EO	
FO	
GO	
HO	
IO	
JO	
KO	

6. What do you notice about the length of each line segment?
7. What is the general term given to this type of line segment?
8. If the point $P(x; y)$ lies on the circle, use the distance formula to determine an expression for the length of PO.
9. Can you deduce a general equation for a circle with centre at the origin?

A circle is the set of all points that are an equal distance (radius) from a given point (centre). In other words, every point on the circumference of a circle is equidistant from its centre.

The radius of a circle is the distance from the centre of a circle to any point on the circumference.

A diameter of a circle is any line passing through the centre of the circle which connects two points on the circle. The diameter is also the name given to the maximum distance between two points on a circle.

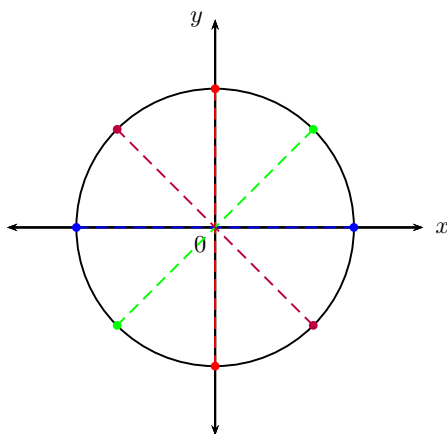
Consider a point $P(x;y)$ on the circumference of a circle of radius r with centre at $O(0;0)$.

In $\triangle OPQ$:

$$OP^2 = PQ^2 + OQ^2 \quad \text{(Pythagoras)}$$
$$OP = r$$
$$PQ = y - 0$$
$$OQ = x - 0$$
$$r^2 = (y-0)^2 + (x-0)^2$$
$$\therefore r^2 = x^2 + y^2$$

Equation of a circle with centre at the origin:
If $P(x;y)$ is a point on a circle with centre $O(0;0)$ and radius r, then the equation of the circle is:

$$x^2 + y^2 = r^2$$

Circle symmetry

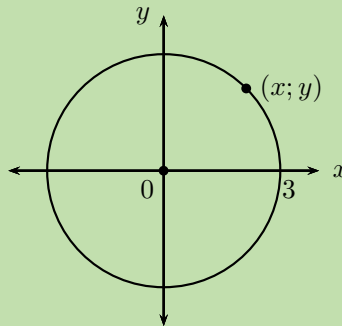
A circle with centre $(0;0)$ is symmetrical about the origin: for every point $(x;y)$ on the circumference of a circle, there is also the point $(-x;-y)$.

A circle centred on the origin is also symmetrical about the x- and y-axis. Is a circle centred on the origin symmetrical about the lines $y = x$ and $y = -x$? How many lines of symmetry does a circle have?

Worked example 4: Equation of a circle with centre at the origin

QUESTION

Given: circle with centre $O(0;0)$ and a radius of 3 units.

1. Sketch the circle on the Cartesian plane.
2. Determine the equation of the circle.
3. Show that the point $T\left(-\sqrt{4};\sqrt{5}\right)$ lies on the circle.

SOLUTION

Step 1: Draw a sketch

Step 2: Determine the equation of the circle

Write down the general form of the equation of a circle with centre $(0;0)$:

$$
\begin{aligned}
& x^2+y^2=r^2 \\
\text{Substitute } r=3: \quad & x^2+y^2=(3)^2 \\
& x^2+y^2=9
\end{aligned}
$$

Step 3: Show that point T lies on the circle

Substitute the x-coordinate and the y-coordinate into the left-hand side of the equation and show that it is equal to the right-hand side:

$$
\begin{aligned}
\text{LHS} &= x^2+y^2 \\
&= \left(-\sqrt{4}\right)^2+\left(\sqrt{5}\right)^2 \\
&= 4+5 \\
&= 9 \\
&= r^2 \\
&= \text{RHS}
\end{aligned}
$$

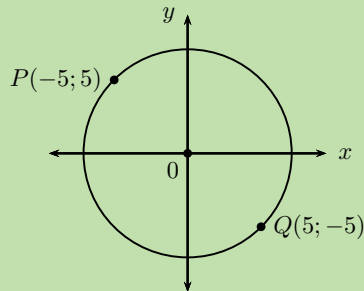
Therefore, $T\left(-\sqrt{4};\sqrt{5}\right)$ lies on the circle $x^2+y^2=9$.

Worked example 5: Equation of a circle with centre at the origin

QUESTION

A circle with centre $O(0;0)$ passes through the points $P(-5;5)$ and $Q(5;-5)$.

1. Plot the points and draw a rough sketch of the circle.
2. Determine the equation of the circle.
3. Calculate the length of PQ.
4. Explain why PQ is a diameter of the circle.

SOLUTION

Step 1: Draw a sketch

Step 2: Determine the equation of the circle

Write down the general form of the equation of a circle with centre $(0;0)$ and substitute $P(-5;5)$:

$$\begin{aligned} x^2 + y^2 &= r^2 \\ (-5)^2 + (5)^2 &= r^2 \\ 25 + 25 &= r^2 \\ 50 &= r^2 \\ \therefore r &= \sqrt{50} \quad (r \text{ is always positive}) \\ r &= 5\sqrt{2} \text{ units} \end{aligned}$$

Therefore, the equation of the circle passing through P and Q is $x^2 + y^2 = 50$.

Step 3: Calculate the length PQ

Use the distance formula to determine the distance between the two points.

$$\begin{aligned} PQ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(5 - (-5))^2 + (-5 - 5)^2} \\ &= \sqrt{(10)^2 + (-10)^2} \\ &= \sqrt{100 + 100} \\ &= \sqrt{100 \ . \ 2} \\ &= 10\sqrt{2} \text{ units} \end{aligned}$$

Step 4: Determine if PQ is a diameter of the circle

$$
\begin{aligned}
r &= 5\sqrt{2} \\
\text{And } d &= 2 \times r \\
&= 2 \times 5\sqrt{2} \\
&= 10\sqrt{2} \\
\therefore d &= PQ
\end{aligned}
$$

Since PQ connects two points, P and Q, on the circle and is a line of length $10\sqrt{2}$ units, PQ is a diameter of the circle.

Alternative method 1**:** Use symmetry to show that PQ is a diameter of the circle.

The diameter is the name given to the maximum distance between two points on a circle; this means that the two points must lie opposite each other with respect to the centre.

Using symmetry about the origin, we know that $(x;y)$ lies opposite $(-x;-y)$ on the circle and vice versa:

- $P(-5;5)$ lies opposite $(5;-5)$, which are the coordinates of Q
- $Q(5;-5)$ lies opposite $(-5;5)$, which are the coordinates of P

Therefore, PQ is a diameter of the circle.

Alternative method 2**:** Show that PQ passes through the centre.

Determine the equation of PQ and show that it passes through the origin.

$$
\begin{aligned}
\frac{y - y_Q}{x - x_Q} &= \frac{y_P - y_Q}{x_P - x_Q} \\
\frac{y+5}{x-5} &= \frac{5+5}{-5-5} \\
\frac{y+5}{x-5} &= \frac{10}{-10} \\
y + 5 &= -(x-5) \\
y &= -x + 5 - 5 \\
y &= -x
\end{aligned}
$$

$\therefore (0;0)$ lines on the line PQ.

Therefore, PQ passes through the centre and is a diameter of the circle.

Worked example 6: Equation of a circle with centre at the origin

QUESTION

Given a circle with centre $O(0;0)$ and a radius of $\sqrt{45}$ units. Determine the possible coordinates of the point(s) on the circle which have an x-value that is twice the y-value.

SOLUTION

Step 1: Determine the equation of the circle

$$\begin{aligned} x^2 + y^2 &= r^2 \\ x^2 + y^2 &= \left(\sqrt{45}\right)^2 \\ x^2 + y^2 &= 45 \end{aligned}$$

Step 2: Determine the coordinates of the points on the circle

To calculate the possible coordinates of the point(s) on the circle which have an x-value that is twice the y-value, we substitute $x = 2y$ into the equation of the circle:

$$\begin{aligned} x^2 + y^2 &= 45 \\ (2y)^2 + y^2 &= 45 \\ 4y^2 + y^2 &= 45 \\ 5y^2 &= 45 \\ y^2 &= 9 \\ \therefore y &= \pm 3 \end{aligned}$$

$$\begin{aligned} \text{If } y = -3: \quad x &= 2(-3) \\ &= -6 \\ \text{If } y = 3: \quad x &= 2(3) \\ &= 6 \end{aligned}$$

This gives the points $(6;3)$ and $(-6;-3)$.

Note: we can check that both points lie on the circle by substituting the coordinates into the equation of the circle:

$$\begin{aligned} (6)^2 + (3)^2 &= 36 + 9 = 45 \\ (-6)^2 + (-3)^2 &= 36 + 9 = 45 \end{aligned}$$

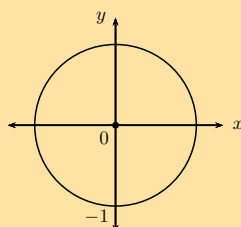
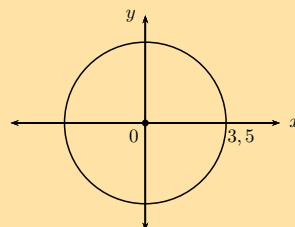
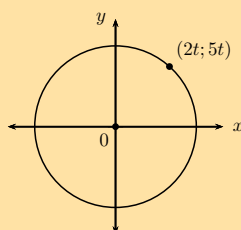
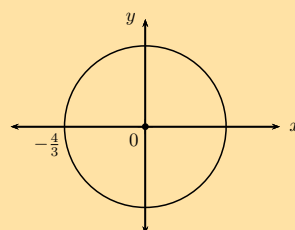
Exercise 7 – 3: Equation of a circle with centre at the origin

1. Complete the following for each circle given below:
    - Determine the radius.
    - Draw a sketch.
    - Calculate the coordinates of two points on the circle.

    a) $x^2 + y^2 = 16$
    b) $x^2 + y^2 = 100$
    c) $3x^2 + 3y^2 = 27$
    d) $y^2 = 20 - x^2$
    e) $x^2 + y^2 = 2{,}25$
    f) $y^2 = -x^2 + \frac{10}{9}$

2. Determine the equation of the circle:

   a) with centre at the origin and a radius of 5 units.
   b) with centre at $(0;0)$ and $r = \sqrt{11}$ units.
   c) passing through the point $(3;5)$ and with centre $(0;0)$.
   d) centred at the origin and $r = 2{,}5$ units.
   e) with centre at the origin and a diameter of 30 units.
   f) passing through the point $(p;3q)$ and with centre at the origin.

   g)

   i)

   h)

   j)

3. Determine whether or not the following equations represent a circle:

   a) $x^2 + y^2 - 8 = 0$
   b) $y^2 - x^2 + 25 = 0$
   c) $3x^2 + 6y^2 = 18$
   d) $x^2 = \sqrt{6} - y^2$
   e) $y(y + x) = -x(x - y) + 11$
   f) $\sqrt{80} + x^2 - y^2 = 0$
   g) $\frac{y^2}{3} + \frac{x^2}{3} = 3$

4. Determine the value(s) of g if $\left(\sqrt{3};g\right)$ is a point on the circle $x^2 + y^2 = 19$.
5. $A(s;t)$ is a point on the circle with centre at the origin and a diameter of 40 cm.

   a) Determine the possible coordinates of A if the value of s is triple the value of t.
   b) Determine the possible coordinates of A if the value of s is half the value of t.

6. $P(-2;3)$ lies on a circle with centre at $(0;0)$.

   a) Determine the equation of the circle.
   b) Sketch the circle and label point P.
   c) If PQ is a diameter of the circle, determine the coordinates of Q.
   d) Calculate the length of PQ.
   e) Determine the equation of the line PQ.
   f) Determine the equation of the line perpendicular to PQ and passing through the point P.

7. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 294S 1b. 294T 1c. 294V 1d. 294W 1e. 294X 1f. 294Y
2a. 294Z 2b. 2952 2c. 2953 2d. 2954 2e. 2955 2f. 2956
2g. 2957 2h. 2958 2i. 2959 2j. 295B 3a. 295C 3b. 295D
3c. 295F 3d. 295G 3e. 295H 3f. 295J 3g. 295K 4. 295M
5. 295N 6. 295P

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Equation of a circle with centre at $(a; b)$ EMCHV

Investigation: Shifting the centre of a circle

Complete the following for each equation in the table below:

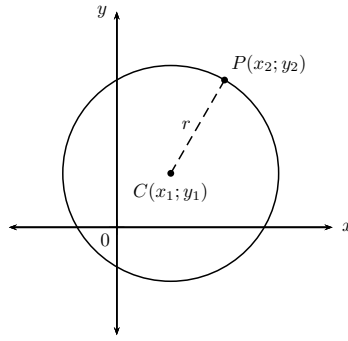
- Write down the shifted equation (do not simplify).
- Draw a rough sketch to illustrate the shift(s).

1. Vertical shift: the graph is shifted 1 unit up.
2. Horizontal shift: the graph is shifted 2 units to the right.
3. Combined shifts: the graph is shifted 1 unit up and 2 units to the right.

The first example has been completed.

Equation	Vertical shift	Horizontal shift	Combined shift
$y - 3x^2 = 0$	$(y - 1) - 3x^2 = 0$	$y - 3(x - 2)^2 = 0$	$(y-1)-3(x-2)^2 = 0$
$y = 3x^2$	$y - 1 = 3x^2$	$y - 3(x - 2)^2 = 0$	$(y - 1) - 3(x - 2)^2 = 0$
$y - 5^x = 0$			
$x^2 + y^2 = 4$			

Use the table to answer the following questions:

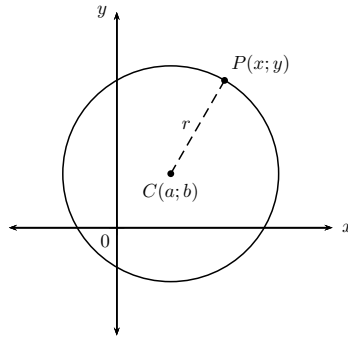
1. Write down the general equation of a circle with centre $(0; 0)$.
2. Write down the general equation of a circle with centre $(0; b)$.
3. Write down the general equation of a circle with centre $(a; 0)$.
4. Write down the general equation of a circle with centre $(a; b)$.

Consider a circle in the Cartesian plane with centre at $C(x_1;y_1)$ and with a radius of r units. If $P(x_2;y_2)$ is any point on the circumference of the circle, we can use the distance formula to calculate the distance between the two points:

$$PC = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The distance PC is equal to the radius (r) of the circle:

$$r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$\therefore r^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

If the coordinates of the centre of the circle are $(a;b)$, then the equation of a circle not centred on the origin is:

$$(x_2 - a)^2 + (y_2 - b)^2 = r^2$$

Equation of a circle with centre at $(a;b)$:
If $P(x;y)$ is a point on a circle with centre $C(a;b)$ and radius r, then the equation of the circle is:

$$(x - a)^2 + (y - b)^2 = r^2$$

A circle with centre $(0;0)$ is a special case of the general equation:

$$(x - a)^2 + (y - b)^2 = r^2$$

$$(x - 0)^2 + (y - 0)^2 = r^2$$
$$\therefore x^2 + y^2 = r^2$$

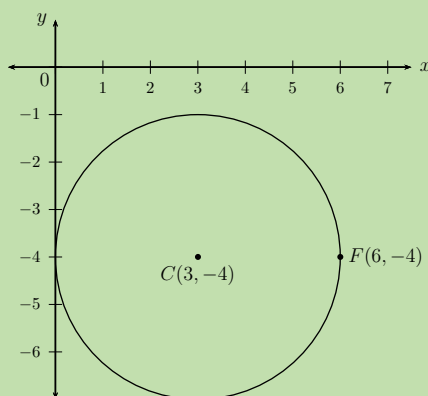
Worked example 7: Equation of a circle with centre at $(a;b)$

QUESTION

$F(6;-4)$ is a point on the circle with centre $(3;-4)$.

1. Draw a rough sketch of the circle and label F.
2. Determine the equation of the circle.
3. Does the point $G\left(\frac{3}{2};-2\right)$ lie on the circle?
4. Does the circle cut the y-axis? Motivate your answer.

SOLUTION

Step 1: Draw a sketch

Step 2: Determine the equation of the circle

Write down the general equation of a circle with centre $(a;b)$ and substitute the coordinates $(3;-4)$:

$$\begin{aligned}(x-a)^2+(y-b)^2&=r^2\\(x-3)^2+(y-(-4))^2&=r^2\\(x-3)^2+(y+4)^2&=r^2\end{aligned}$$

Substitute the coordinates of $F(6;-4)$ to determine the value of r^2:

$$\begin{aligned}(6-3)^2+(-4+4)^2&=r^2\\(3)^2+(0)^2&=r^2\\9&=r^2\\\therefore (x-3)^2+(y+4)^2&=9\end{aligned}$$

Step 3: Determine whether or not G lies on the circle

If $G\left(\frac{3}{2};-2\right)$ lies on the circle, then it will satisfy the equation of the circle:

$$\begin{aligned}
\text{LHS} &= \left(\frac{3}{2}-3\right)^2+(-2+4)^2 \\
&= \left(-\frac{3}{2}\right)^2+(2)^2 \\
&= \frac{9}{4}+4 \\
&= \frac{9}{4}+\frac{16}{4} \\
&= \frac{25}{4} \\
\\
\text{RHS} &= 9 \\
\therefore \text{LHS} &\neq \text{RHS}
\end{aligned}$$

Therefore G does not lie on the circle.

Step 4: Determine the y-intercept(s)

To determine the y-intercept(s), we let $x=0$:

$$\begin{aligned}
(0-3)^2+(y+4)^2 &= 9 \\
9+(y+4)^2 &= 9 \\
(y+4)^2 &= 0 \\
y+4 &= 0 \\
\therefore y &= -4
\end{aligned}$$

The circle cuts the y-axis at $(0;-4)$.

Worked example 8: Equation of a circle with centre at $(a;b)$

QUESTION

Determine the coordinates of the centre of the circle and the length of the radius for $3x^2+6x+3y^2-12y-33=0$.

SOLUTION

Step 1: Make the coefficient of the x^2 term and the y^2 term equal to 1

The coefficient of the x^2 and y^2 term must be 1, so we take out 3 as a common factor:

$$x^2+2x+y^2-4y-11=0$$

Step 2: Complete the square

Take half the **coefficient of the x term**, square it; then add and subtract it from the equation.

The coefficient of the x term is 2, so then $\left(\frac{2}{2}\right)^2=(1)^2=1$.

Take half the **coefficient of the y term**, square it; then add and subtract it from the equation.

The coefficient of the y term is -4, so then $\left(\frac{-4}{2}\right)^2 = (-2)^2 = 4$.

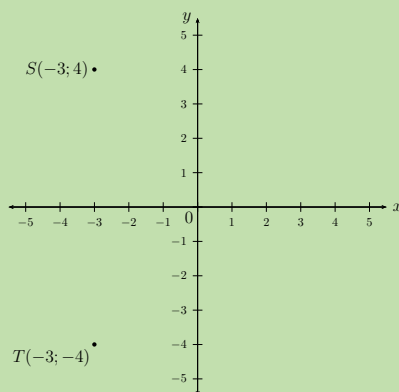
$$\begin{aligned} x^2 + 2x + y^2 - 4y - 11 &= 0 \\ (x^2 + 2x + 1) - 1 + (y^2 - 4y + 4) - 4 - 11 &= 0 \\ (x+1)^2 + (y-2)^2 - 16 &= 0 \\ (x+1)^2 + (y-2)^2 &= 16 \end{aligned}$$

The centre of the circle is $(-1; 2)$ and the radius is 4 units.

See video: 293G at www.everythingmaths.co.za

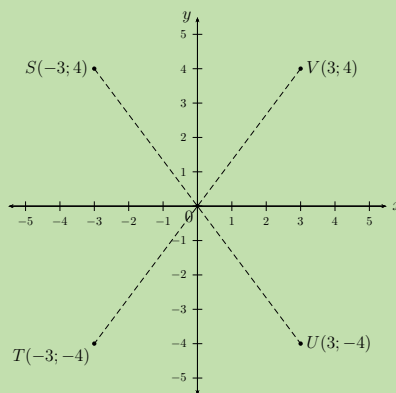
Worked example 9: Equation of a circle with centre at $(a; b)$

QUESTION

Given $S(-3; 4)$ and $T(-3; -4)$ on the Cartesian plane.

1. The points U and V are symmetrical about the origin to S and T respectively. Determine the coordinates of U and V.
2. Determine the mid-point of SU.
3. Write down the equation of the circle $STUV$.
4. Is $S\hat{T}U = 90°$? Give reasons.
5. Determine the equation of the line perpendicular to SU and passing through point S.

SOLUTION

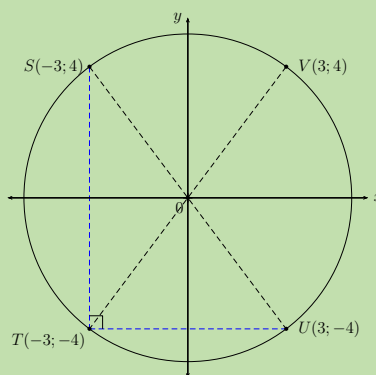
Step 1: Determine the coordinates of U and V

For symmetry about the origin, every point $(x; y)$ is symmetrical to $(-x; -y)$. So $S(-3; 4)$ is symmetrical to $U(3; -4)$ and $T(-3; -4)$ is symmetrical to $V(3; 4)$.

Step 2: Determine the mid-point of SU

$$
\begin{aligned}
M(x;y) &= \left(\frac{x_U + x_S}{2}; \frac{y_U + y_S}{2}\right) \\
&= \left(\frac{3-3}{2}; \frac{-4+4}{2}\right) \\
&= (0;0)
\end{aligned}
$$

The mid-point of the line SU is the origin.

Step 3: Determine the equation of the circle

$$
\begin{aligned}
x^2 + y^2 &= r^2 \\
(-3)^2 + (4)^2 &= r^2 \\
9 + 16 &= r^2 \\
25 &= r^2 \\
\therefore x^2 + y^2 &= 25
\end{aligned}
$$

SU passes through the centre of the circle and is therefore a diameter. From Euclidean geometry, we know that the diameter of the circle subtends a right-angle at the circumference, therefore $S\hat{T}U = 90°$ (angle in semi-circle).

Step 4: Determine the equation of the line perpendicular to SU at point S

Determine the gradient of SU:

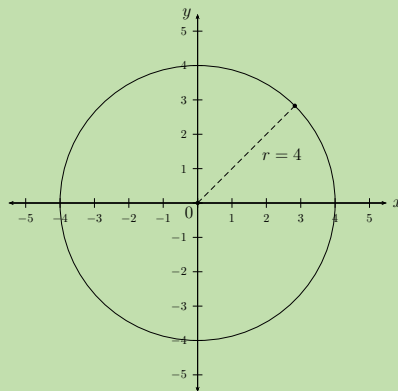
$$\begin{aligned} m_{SU} &= \frac{y_S - y_U}{x_S - x_U} \\ &= \frac{4-(-4)}{-3-3} \\ &= \frac{8}{-6} \\ &= -\frac{4}{3} \end{aligned}$$

Let the gradient of the line perpendicular to SU be m_P:

$$\begin{aligned} m_{SU} \times m_P &= -1 \\ -\frac{4}{3} \times m_P &= -1 \\ \therefore m_P &= \frac{3}{4} \\ y - y_1 &= m(x - x_1) \\ \text{Substitute } S(-3;4): \quad y - 4 &= \frac{3}{4}(x-(-3)) \\ y - 4 &= \frac{3}{4}(x+3) \\ y &= \frac{3}{4}x + \frac{25}{4} \end{aligned}$$

Worked example 10: Equation of a circle with centre at $(a;b)$

QUESTION

Given a circle with centre $(0;0)$ and a radius of 4 units.

1. If the circle is shifted 2 units down and 1 unit to the right, write down the equation of the shifted circle.
2. Sketch the original circle and the shifted circle on the same system of axes.

3. The shifted circle is reflected about the line $y = x$. Sketch the reflected circle on the same system of axes as the question above.
4. Write down the equation of the reflected circle.

SOLUTION

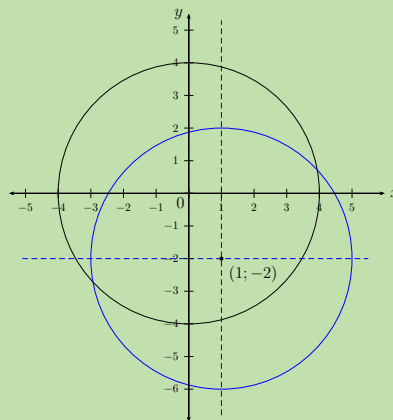
Step 1: Write down the equation of the circle

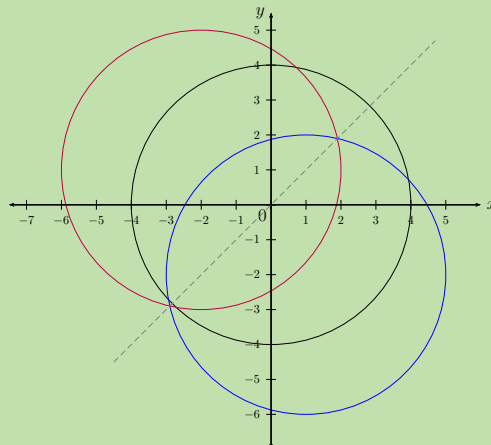
$$x^2 + y^2 = 16$$

Step 2: Determine the equation of the shifted circle

- Vertical shift: 2 units down: y is replaced with $y + 2$
- Horizontal shift: 1 unit to the right: x is replaced with $x - 1$

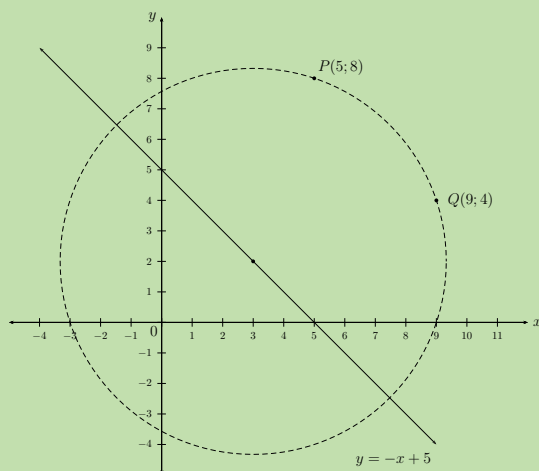
Therefore the equation of the shifted circle is $(x - 1)^2 + (y + 2)^2 = 16$ with centre at $(1; -2)$ and radius of 4 units.

Step 3: Draw a sketch of the circles

The shifted circle is reflected about the line $y = x$. The x and y variables are interchanged to give the circle with equation $(y-1)^2 + (x+2)^2 = 16$ and centre at $(-2; 1)$.

Worked example 11: Equation of a circle with centre at $(a; b)$

QUESTION

A circle with centre on the line $y = -x + 5$ passes through the points $P(5; 8)$ and $Q(9; 4)$. Determine the equation of the circle.

SOLUTION

Step 1: Draw a rough sketch

Step 2: Write down the general equation of a circle

$$(x - a)^2 + (y - b)^2 = r^2$$

Consider the line $y = -x+5$. Any point on this line will have the coordinates $(x; -x+5)$. Since the centre of the circle lies on the line $y = -x+5$, we can write the equation of the circle as

$$\begin{aligned}(x-a)^2 + (y - (-a+5))^2 &= r^2 \\ (x-a)^2 + (y+a-5)^2 &= r^2\end{aligned}$$

Step 3: Solve for the unknown variables a and r

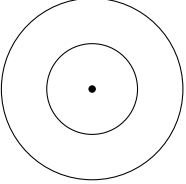
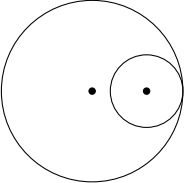
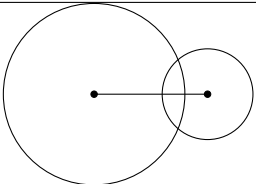
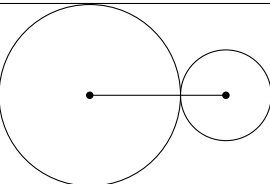
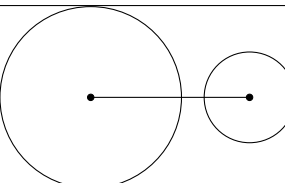
We need two equations to solve for the two unknown variables. We substitute the two given points, $P(5; 8)$ and $Q(9; 4)$ and solve for a and r simultaneously:

$$\begin{aligned}\text{Substitute } P(5;8): \quad (5-a)^2 + (8+a-5)^2 &= r^2 \\ (5-a)^2 + (a+3)^2 &= r^2 \\ 25 - 10a + a^2 + a^2 + 6a + 9 &= r^2 \\ 2a^2 - 4a + 34 &= r^2 \ldots (1)\end{aligned}$$

$$\begin{aligned}\text{Substitute } Q(9;4): \quad (9-a)^2 + (4+a-5)^2 &= r^2 \\ (9-a)^2 + (a-1)^2 &= r^2 \\ 81 - 18a + a^2 + a^2 - 2a + 1 &= r^2 \\ 2a^2 - 20a + 82 &= r^2 \ldots (2)\end{aligned}$$

$$\begin{aligned}(1)-(2):\quad 16a-48&=0\\ 16a&=48\\ \therefore a&=3\end{aligned}$$

$$\begin{aligned}\text{Substitute into }(2):\quad r^2&=2(3)^2-20(3)+82\\ &=18-60+82\\ &=40\end{aligned}$$

$$\begin{aligned}\text{And}\quad b&=-a+5\\ &=-3+5\\ &=2\end{aligned}$$

Step 4: Write the final answer

The equation of the circle is $(x-3)^2+(y-2)^2=40$.

	Circles with the same centre.
	Circles touch internally at one point: the distance between the centres is equal to the difference between the radii. $d=r_2-r_1$
	Circles intersect at two points: the distance between the centres is less than the sum of the radii. $d<r_1+r_2$
	Circles touch externally at one point: the distance between the centres is equal to the sum of the radii. $d=r_1+r_2$
	Circles do not touch or intersect: the distance between the centres is greater than the sum of the radii. $d>r_1+r_2$

Exercise 7 – 4: Equation of a circle with centre at $(a; b)$

1. Determine whether or not each of the following equations represents a circle. If not, give a reason.

   a) $x^2 + y^2 + 6y - 10 = 0$
   b) $3x^2 - 35 + 3y^2 = 9y$
   c) $40 = x^2 + 2x + 4y^2$
   d) $x^2 - 4x = \sqrt{21} + 5y + y^2$
   e) $3\sqrt{7} - x^2 - y^2 + 6y - 8x = 0$
   f) $(x - 1)^2 + (y + 2)^2 + 9 = 0$

2. Write down the equation of the circle:

   a) with centre $(0; 4)$ and a radius of 3 units.
   b) such that $r = 5$ and the centre is the origin.
   c) with centre $(-2; 3)$ and passing through the point $(4; 5)$.
   d) with centre $(p; -q)$ and $r = \sqrt{6}$.
   e) with $r = \sqrt{10}$ and centre $\left(-\frac{1}{2}; \frac{3}{2}\right)$ and
   f) with centre $(1; -5)$ and passing through the origin.

3. Determine the centre and the length of the radius for the following circles:

   a) $x^2 = 21 - y^2 + 4y$
   b) $y^2 + x + x^2 - \frac{15}{4} = 0$
   c) $x^2 - 4x + y^2 + 2y - 5 = 0$
   d) $x^2 + y^2 - 6y + 2x - 15 = 0$
   e) $5 - x^2 - 6x - 8y - y^2 = 0$
   f) $x^2 - \frac{2}{3}x + y^2 - 4y = \frac{35}{9}$
   g) $16x + 2y^2 - 20y + 2x^2 + 42 = 0$
   h) $6x - 6y - x^2 - y^2 = 6$

4. A circle cuts the x-axis at $R(-2; 0)$ and $S(2; 0)$. If $r = \sqrt{20}$ units, determine the possible equation(s) of the circle. Draw a sketch.
5. $P(1; 2)$ and $Q(-5; -6)$ are points on a circle such that PQ is a diameter. Determine the equation of the circle.
6. A circle with centre $N(4; 4)$ passes through the points $K(1; 6)$ and $L(6; 7)$.

   a) Determine the equation of the circle.
   b) Determine the coordinates of M, the mid-point of KL.
   c) Show that $MN \perp KL$.
   d) If P is a point on the circle such that LP is a diameter, determine the coordinates of P.
   e) Determine the equation of the line LP.

7. A circle passes through the point $A(7; -4)$ and $B(-5; -2)$. If its centre lies on the line $y + 5 = 2x$, determine the equation of the circle.
8. A circle with centre $(0; 0)$ passes through the point $T(3; 5)$.

   a) Determine the equation of the circle.
   b) If the circle is shifted 2 units to the right and 3 units down, determine the new equation of the circle.
   c) Draw a sketch of the original circle and the shifted circle on the same system of axes.
   d) On the same system of axes as the previous question, draw a sketch of the shifted circle reflected about the x-axis. State the coordinates of the centre of this circle.

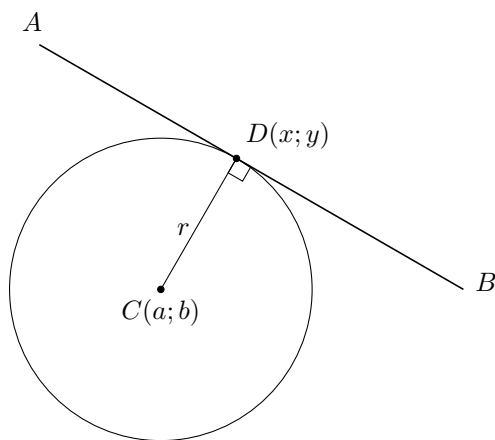
9. Determine whether the circle $x^2 - 4x + y^2 - 6y + 9 = 0$ cuts, touches or does not intersect the x-axis and the y-axis.
10. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 295Q	1b. 295R	1c. 295S	1d. 295T	1e. 295V	1f. 295W
2a. 295X	2b. 295Y	2c. 295Z	2d. 2962	2e. 2963	2f. 2964
3a. 2965	3b. 2966	3c. 2967	3d. 2968	3e. 2969	3f. 296B
3g. 296C	3h. 296D	4. 296F	5. 296G	6. 296H	7. 296J
8. 296K	9. 296M				

7.3 Equation of a tangent to a circle EMCHW

Investigation:

1. On a suitable system of axes, draw the circle $x^2 + y^2 = 20$ with centre at $O(0;0)$.
2. Plot the point $T(2;4)$.
3. Plot the point $P(0;5)$. Draw PT and extend the line so that is cuts the positive x-axis.
4. Measure $O\hat{T}P$.
5. Determine the gradient of the radius OT.
6. Determine the gradient of PT.
7. Prove that $PT \perp OT$.
8. Plot the point $S(2;-4)$ and join OS.
9. Draw a tangent to the circle at S.
10. Measure the angle between OS and the tangent line at S.
11. Make a conjecture about the angle between the radius and the tangent to a circle at a point on the circle.
12. Complete the sentence: the product of the ...... of the radius and the gradient of the ...... is equal to ....

A circle with centre $C(a;b)$ and a radius of r units is shown in the diagram above. $D(x;y)$ is a point on the circumference and the equation of the circle is:

$$(x-a)^2 + (y-b)^2 = r^2$$

A tangent is a straight line that touches the circumference of a circle at only one place.

The tangent line AB touches the circle at D.

The radius of the circle CD is perpendicular to the tangent AB at the point of contact D.

$$CD \perp AB$$
$$\text{and } C\hat{D}A = C\hat{D}B = 90^\circ$$

The product of the gradient of the radius and the gradient of the tangent line is equal to -1.

$$m_{CD} \times m_{AB} = -1$$

How to determine the equation of a tangent:

1. Determine the equation of the circle and write it in the form

$$(x-a)^2 + (y-b)^2 = r^2$$

2. From the equation, determine the coordinates of the centre of the circle $(a;b)$.
3. Determine the gradient of the radius:

$$m_{CD} = \frac{y_2 - y_1}{x_2 - x_1}$$

4. The radius is perpendicular to the tangent of the circle at a point D so:

$$m_{AB} = -\frac{1}{m_{CD}}$$

5. Write down the gradient-point form of a straight line equation and substitute m_{AB} and the coordinates of D. Make y the subject of the equation.

$$y - y_1 = m(x - x_1)$$

Worked example 12: Equation of a tangent to a circle

QUESTION

Determine the equation of the tangent to the circle $x^2 + y^2 - 2y + 6x - 7 = 0$ at the point $F(-2; 5)$.

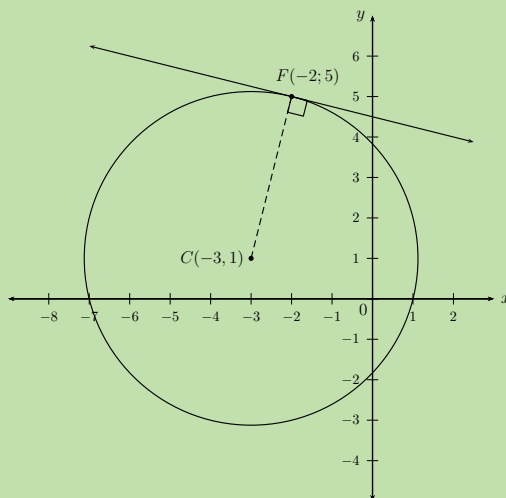
SOLUTION

Step 1: Write the equation of the circle in the form $(x-a)^2 + (y-b)^2 = r^2$

Use the method of completing the square:

$$\begin{aligned} x^2 + y^2 - 2y + 6x - 7 &= 0 \\ x^2 + 6x + y^2 - 2y &= 7 \\ (x^2 + 6x + 9) - 9 + (y^2 - 2y + 1) - 1 &= 7 \\ (x+3)^2 + (y-1)^2 &= 17 \end{aligned}$$

Step 2: Draw a sketch

The centre of the circle is $(-3; 1)$ and the radius is $\sqrt{17}$ units.

Step 3: Determine the gradient of the radius CF

$$\begin{aligned} m_{CF} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{5-1}{-2+3} \\ &= 4 \end{aligned}$$

Step 4: Determine the gradient of the tangent

Let the gradient of the tangent line be m.

$$\begin{aligned} m_{CF} \times m &= -1 \\ 4 \times m &= -1 \\ \therefore m &= -\frac{1}{4} \end{aligned}$$

Step 5: Determine the equation of the tangent to the circle

Write down the gradient-point form of a straight line equation and substitute $m = -\frac{1}{4}$ and $F(-2;5)$.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - y_1 &= -\frac{1}{4}(x - x_1) \\ \text{Substitute } F(-2;5): \quad y - 5 &= -\frac{1}{4}(x - (-2)) \\ y - 5 &= -\frac{1}{4}(x + 2) \\ y &= -\frac{1}{4}x - \frac{1}{2} + 5 \\ &= -\frac{1}{4}x + \frac{9}{2} \end{aligned}$$

Step 6: Write the final answer

The equation of the tangent to the circle at F is $y = -\frac{1}{4}x + \frac{9}{2}$.

Worked example 13: Equation of a tangent to a circle

QUESTION

The straight line $y = x + 4$ cuts the circle $x^2 + y^2 = 26$ at P and Q.

1. Calculate the coordinates of P and Q.
2. Sketch the circle and the straight line on the same system of axes. Label points P and Q.
3. Determine the coordinates of H, the mid-point of chord PQ.
4. If O is the centre of the circle, show that $PQ \perp OH$.
5. Determine the equations of the tangents to the circle at P and Q.
6. Determine the coordinates of S, the point where the two tangents intersect.
7. Show that S, H and O are on a straight line.

SOLUTION

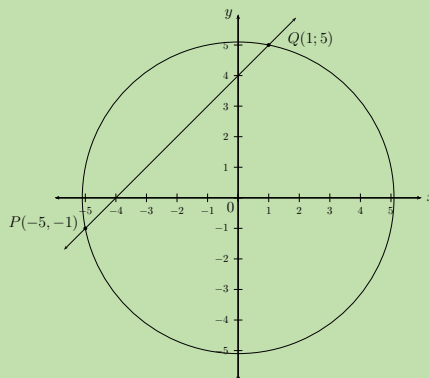
Step 1: Determine the coordinates of P and Q

Substitute the straight line $y = x + 4$ into the equation of the circle and solve for x:

$$\begin{aligned} x^2 + y^2 &= 26 \\ x^2 + (x + 4)^2 &= 26 \\ x^2 + x^2 + 8x + 16 &= 26 \\ 2x^2 + 8x - 10 &= 0 \\ x^2 + 4x - 5 &= 0 \\ (x - 1)(x + 5) &= 0 \\ \therefore x &= 1 \text{ or } x = -5 \end{aligned}$$

$$\begin{aligned} \text{If } x = 1 \quad & y = 1 + 4 = 5 \\ \text{If } x = -5 \quad & y = -5 + 4 = -1 \end{aligned}$$

This gives the points $P(-5;-1)$ and $Q(1;5)$.

Step 2: Draw a sketch

Step 3: Determine the coordinates of the mid-point H

$$\begin{aligned} H(x;y) &= \left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right) \\ &= \left(\frac{1-5}{2}; \frac{5-1}{2}\right) \\ &= \left(\frac{-4}{2}; \frac{4}{2}\right) \\ &= (-2;2) \end{aligned}$$

Step 4: Show that OH is perpendicular to PQ

We need to show that the product of the two gradients is equal to -1. From the given equation of PQ, we know that $m_{PQ} = 1$.

$$\begin{aligned} m_{OH} &= \frac{2-0}{-2-0} \\ &= -1 \\ m_{PQ} \times m_{OH} &= -1 \\ \therefore PQ &\perp OH \end{aligned}$$

Step 5: Determine the equations of the tangents at P and Q

Tangent at P:

Determine the gradient of the radius OP:

$$\begin{aligned} m_{OP} &= \frac{-1-0}{-5-0} \\ &= \frac{1}{5} \end{aligned}$$

The tangent of a circle is perpendicular to the radius, therefore we can write:

$$\begin{aligned} \frac{1}{5} \times m_P &= -1 \\ \therefore m_P &= -5 \end{aligned}$$

Substitute $m_P = -5$ and $P(-5;-1)$ into the equation of a straight line.

$$y - y_1 = -5(x - x_1)$$

Substitute $P(-5;-1)$:

$$\begin{aligned} y + 1 &= -5(x+5) \\ y &= -5x - 25 - 1 \\ &= -5x - 26 \end{aligned}$$

Tangent at Q:

Determine the gradient of the radius OQ:

$$\begin{aligned} m_{OQ} &= \frac{5-0}{1-0} \\ &= 5 \end{aligned}$$

The tangent of a circle is perpendicular to the radius, therefore we can write:

$$\begin{aligned} 5 \times m_Q &= -1 \\ \therefore m_Q &= -\frac{1}{5} \end{aligned}$$

Substitute $m_Q = -\frac{1}{5}$ and $Q(1;5)$ into the equation of a straight line.

$$y - y_1 = -\frac{1}{5}(x - x_1)$$

Substitute $Q(1;5)$:

$$\begin{aligned} y - 5 &= -\frac{1}{5}(x-1) \\ y &= -\frac{1}{5}x + \frac{1}{5} + 5 \\ &= -\frac{1}{5}x + \frac{26}{5} \end{aligned}$$

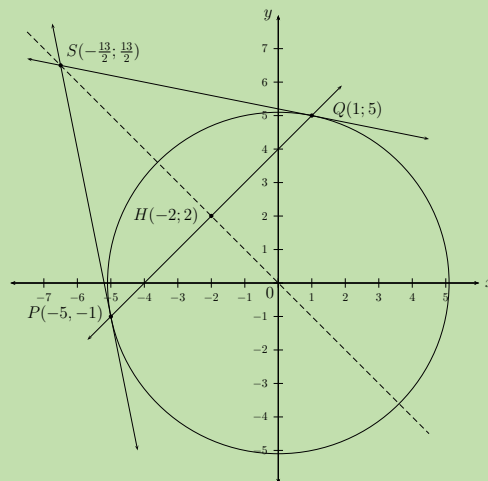
The equations of the tangents are $y = -5x - 26$ and $y = -\frac{1}{5}x + \frac{26}{5}$.

Step 6: Determine the coordinates of S

Equate the two linear equations and solve for x:

$$\begin{aligned} -5x - 26 &= -\frac{1}{5}x + \frac{26}{5} \\ -25x - 130 &= -x + 26 \\ -24x &= 156 \\ x &= -\frac{156}{24} \\ &= -\frac{13}{2} \end{aligned}$$

$$\text{If } x = -\frac{13}{2} \quad y = -5\left(-\frac{13}{2}\right) - 26$$
$$= \frac{65}{2} - 26$$
$$= \frac{13}{2}$$

This gives the point $S\left(-\frac{13}{2}; \frac{13}{2}\right)$.

Step 7: Show that S, H and O are on a straight line

We need to show that there is a constant gradient between any two of the three points. We have already shown that PQ is perpendicular to OH, so we expect the gradient of the line through S, H and O to be -1.

$$\begin{aligned} m_{SH} &= \frac{\frac{13}{2} - 2}{-\frac{13}{2} + 2} \\ &= -1 \end{aligned}$$

$$\begin{aligned} m_{SO} &= \frac{\frac{13}{2} - 0}{-\frac{13}{2} - 0} \\ &= -1 \end{aligned}$$

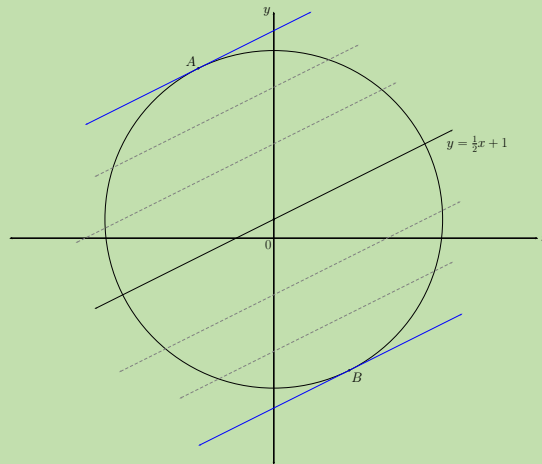
Therefore S, H and O all lie on the line $y = -x$.

Worked example 14: Equation of a tangent to a circle

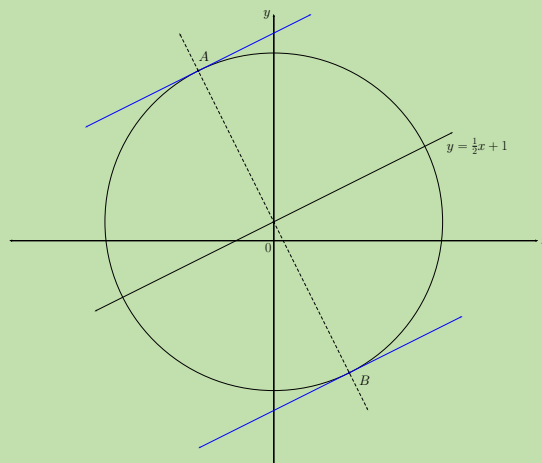
QUESTION

Determine the equations of the tangents to the circle $x^2 + (y-1)^2 = 80$, given that both are parallel to the line $y = \frac{1}{2}x + 1$.

SOLUTION

Step 1: Draw a sketch

The tangents to the circle, parallel to the line $y = \frac{1}{2}x + 1$, must have a gradient of $\frac{1}{2}$. From the sketch we see that there are two possible tangents.

Step 2: Determine the coordinates of A and B

To determine the coordinates of A and B, we must find the equation of the line perpendicular to $y = \frac{1}{2}x + 1$ and passing through the centre of the circle. This perpendicular line will cut the circle at A and B.

$$
\begin{aligned}
y &= \frac{1}{2}x + 1 \\
\therefore m &= \frac{1}{2} \\
m_{\perp} &= -\frac{1}{m} = -2 \\
\therefore y &= -2x + 1
\end{aligned}
$$

Notice that the line passes through the centre of the circle.

To determine the coordinates of A and B, we substitute the straight line $y = -2x + 1$ into the equation of the circle and solve for x:

$$
\begin{aligned}
x^2 + (y-1)^2 &= 80 \\
x^2 + (-2x + 1 - 1)^2 &= 80 \\
x^2 + 4x^2 &= 80 \\
5x^2 &= 80 \\
x^2 &= 16 \\
\therefore x &= \pm 4 \\
\text{If } x = 4 \quad y &= -2(4) + 1 = -7 \\
\text{If } x = -4 \quad y &= -2(-4) + 1 = 9
\end{aligned}
$$

This gives the points $A(-4; 9)$ and $B(4; -7)$.

Step 3: Determine the equations of the tangents to the circle
Tangent at A:

$$
\begin{aligned}
y - y_1 &= \frac{1}{2}(x - x_1) \\
y - 9 &= \frac{1}{2}(x + 4) \\
y &= \frac{1}{2}x + 11
\end{aligned}
$$

Tangent at B:

$$
\begin{aligned}
y - y_1 &= \frac{1}{2}(x - x_1) \\
y + 7 &= \frac{1}{2}(x - 4) \\
y &= \frac{1}{2}x - 9
\end{aligned}
$$

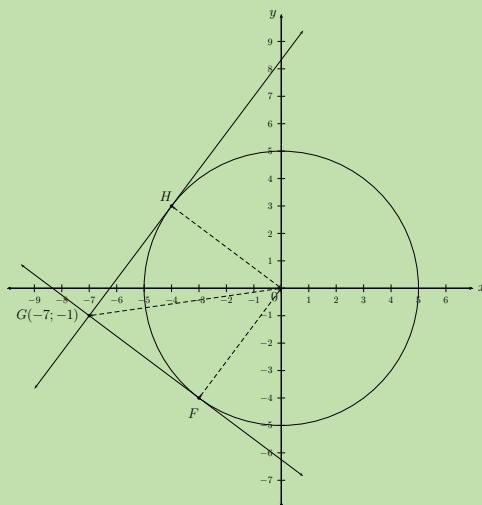
The equation of the tangent at point A is $y = \frac{1}{2}x + 11$ and the equation of the tangent at point B is $y = \frac{1}{2}x - 9$.

Worked example 15: Equation of a tangent to a circle

QUESTION

Determine the equations of the tangents to the circle $x^2 + y^2 = 25$, from the point $G(-7; -1)$ outside the circle.

SOLUTION

Step 1: Draw a sketch

Step 2: Consider where the two tangents will touch the circle

Let the two tangents from G touch the circle at F and H.

$$\begin{aligned} OF = OH &= 5 \text{ units} \quad \text{(equal radii)} \\ OG &= \sqrt{(0+7)^2 + (0+1)^2} \\ &= \sqrt{50} \\ GF &= \sqrt{(x+7)^2 + (y+1)^2} \\ \therefore GF^2 &= (x+7)^2 + (y+1)^2 \\ \text{And } G\hat{F}O = G\hat{H}O &= 90^\circ \end{aligned}$$

Consider $\triangle GFO$ and apply the theorem of Pythagoras:

$$\begin{aligned} GF^2 + OF^2 &= OG^2 \\ (x+7)^2 + (y+1)^2 + 5^2 &= \left(\sqrt{50}\right)^2 \\ x^2 + 14x + 49 + y^2 + 2y + 1 + 25 &= 50 \\ x^2 + 14x + y^2 + 2y + 25 &= 0 \ldots\ldots (1) \\ \text{Substitute } y^2 &= 25 - x^2 \text{ into equation } (1) \\ x^2 + 14x + \left(25 - x^2\right) + 2\left(\sqrt{25 - x^2}\right) + 25 &= 0 \\ 14x + 50 &= -2\left(\sqrt{25 - x^2}\right) \\ 7x + 25 &= -\sqrt{25 - x^2} \end{aligned}$$

$$\text{Square both sides: } (7x+25)^2 = \left(-\sqrt{25-x^2}\right)^2$$
$$\begin{aligned}
49x^2 + 350x + 625 &= 25 - x^2 \\
50x^2 + 350x + 600 &= 0 \\
x^2 + 7x + 12 &= 0 \\
(x+3)(x+4) &= 0 \\
\therefore x = -3 &\text{ or } x = -4 \\
\text{At } F: x = -3 \quad y &= -\sqrt{25-(-3)^2} = -\sqrt{16} = -4 \\
\text{At } H: x = -4 \quad y &= \sqrt{25-(-4)^2} = \sqrt{9} = 3
\end{aligned}$$

Note: from the sketch we see that F must have a negative y-coordinate, therefore we take the negative of the square root. Similarly, H must have a positive y-coordinate, therefore we take the positive of the square root.

This gives the points $F(-3;-4)$ and $H(-4;3)$.

Tangent at F:

$$\begin{aligned}
m_{FG} &= \frac{-1+4}{-7+3} \\
&= -\frac{3}{4}
\end{aligned}$$

$$\begin{aligned}
y - y_1 &= m(x - x_1) \\
y - y_1 &= -\frac{3}{4}(x - x_1) \\
y + 1 &= -\frac{3}{4}(x + 7) \\
y &= -\frac{3}{4}x - \frac{21}{4} - 1 \\
y &= -\frac{3}{4}x - \frac{25}{4}
\end{aligned}$$

Tangent at H:

$$\begin{aligned}
m_{HG} &= \frac{-1-3}{-7+4} \\
&= \frac{4}{3}
\end{aligned}$$

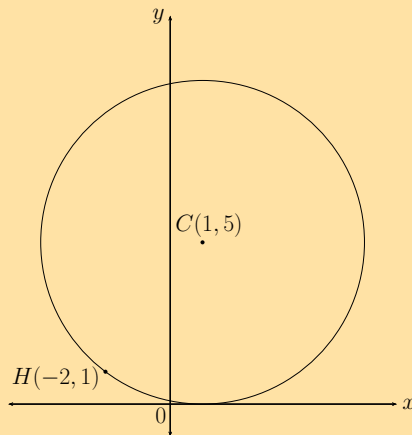
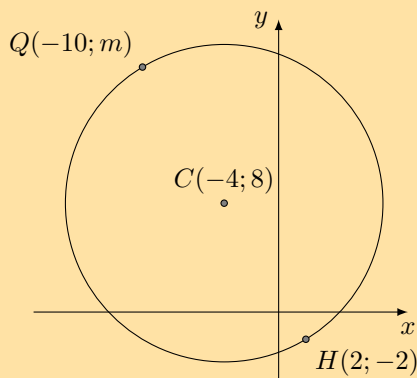
$$\begin{aligned}
y + 1 &= \frac{4}{3}(x + 7) \\
y &= \frac{4}{3}x + \frac{28}{3} - 1 \\
y &= \frac{4}{3}x + \frac{25}{3}
\end{aligned}$$

Step 3: Write the final answer

The equations of the tangents to the circle are $y = -\frac{3}{4}x - \frac{25}{4}$ and $y = \frac{4}{3}x + \frac{25}{3}$.

See video: 293H at www.everythingmaths.co.za

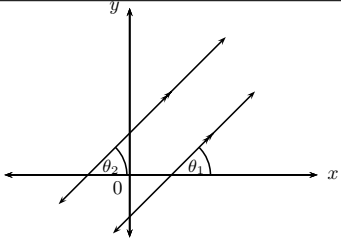
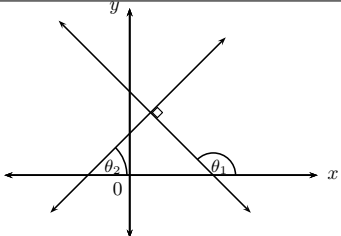
Exercise 7 – 5: Equation of a tangent to a circle

1. a) A circle with centre $(8;-7)$ and the point $(5;-5)$ on the circle are given. Determine the gradient of the radius.
   b) Determine the gradient of the tangent to the circle at the point $(5;-5)$.
2. Given the equation of the circle: $(x+4)^2+(y+8)^2=136$
   a) Find the gradient of the radius at the point $(2;2)$ on the circle.
   b) Determine the gradient of the tangent to the circle at the point $(2;2)$.
3. Given a circle with the central coordinates $(a;b)=(-9;6)$. Determine the equation of the tangent to the circle at the point $(-2;5)$.
4. Given the diagram below:

   Determine the equation of the tangent to the circle with centre C at point H.
5. Given the point $P(2;-4)$ on the circle $(x-4)^2+(y+5)^2=5$. Find the equation of the tangent at P.
6. $C(-4;8)$ is the centre of the circle passing through $H(2;-2)$ and $Q(-10;m)$.

   a) Determine the equation of the circle.
   b) Determine the value of m.
   c) Determine the equation of the tangent to the circle at point Q.

7. The straight line $y = x + 2$ cuts the circle $x^2 + y^2 = 20$ at P and Q.
   a) Calculate the coordinates of P and Q.
   b) Determine the length of PQ.
   c) Determine the coordinates of M, the mid-point of chord PQ.
   d) If O is the centre of the circle, show that $PQ \perp OM$.
   e) Determine the equations of the tangents to the circle at P and Q.
   f) Determine the coordinates of S, the point where the two tangents intersect.
   g) Show that $PS = QS$.
   h) Determine the equations of the two tangents to the circle, both parallel to the line $y + 2x = 4$.
8. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 296N 2. 296P 3. 296Q 4. 296R 5. 296S 6. 296T
7. 296V

 www.everythingmaths.co.za

7.4 Summary EMCHX

See video: 294R at www.everythingmaths.co.za

Theorem of Pythagoras:	$AB^2 = AC^2 + BC^2$
Distance formula:	$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
Gradient:	$m_{AB} = \frac{y_2 - y_1}{x_2 - x_1}$ or $m_{AB} = \frac{y_1 - y_2}{x_1 - x_2}$
Mid-point of a line segment:	$M(x; y) = \left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right)$
Points on a straight line:	$m_{AB} = m_{AM} = m_{MB}$

Straight line equations	Formulae
Two-point form:	$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$
Gradient-point form:	$y - y_1 = m(x - x_1)$
Gradient-intercept form:	$y = mx + c$
Horizontal lines:	$y = k$
Vertical lines:	$x = k$

Parallel lines		$m_1 = m_2$	$\theta_1 = \theta_2$
Perpendicular lines		$m_1 \times m_2 = -1$	$\theta_1 = 90^\circ + \theta_2$

- Inclination of a straight line: the gradient of a straight line is equal to the tangent of the angle formed between the line and the positive direction of the x-axis.

$$m = \tan\theta \qquad \text{for } 0^\circ \leq \theta < 180^\circ$$

- Equation of a circle with centre at the origin:
  If $P(x; y)$ is a point on a circle with centre $O(0; 0)$ and radius r, then the equation of the circle is:

$$x^2 + y^2 = r^2$$

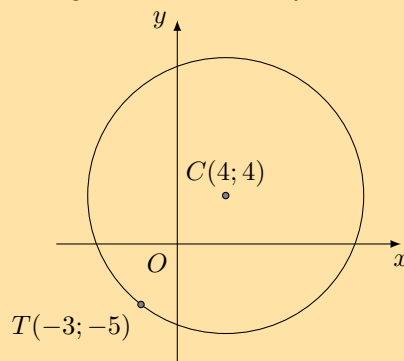
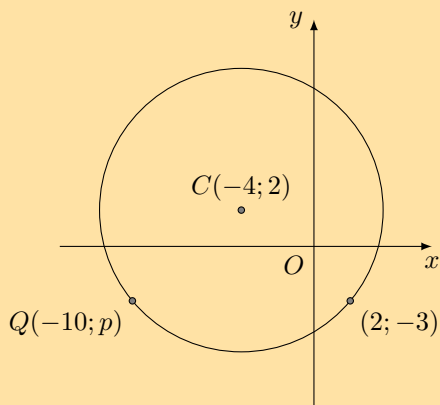
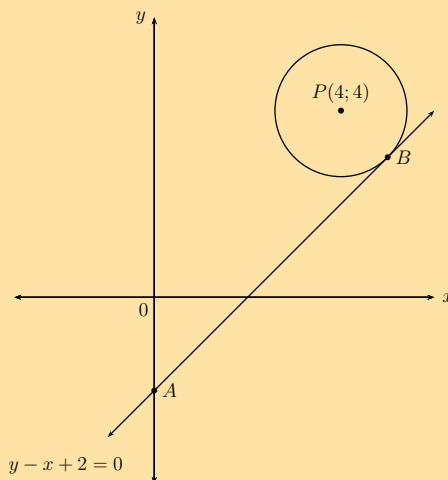
- General equation of a circle with centre at $(a; b)$:
  If $P(x; y)$ is a point on a circle with centre $C(a; b)$ and radius r, then the equation of the circle is:

$$(x - a)^2 + (y - b)^2 = r^2$$

- A tangent is a straight line that touches the circumference of a circle at only one point.

- The radius of a circle is perpendicular to the tangent at the point of contact.

Exercise 7 – 6: End of chapter exercises

1. Find the equation of the circle:
    a) with centre $(0;5)$ and radius 5
    b) with centre $(2;0)$ and radius 4
    c) with centre $(-5;7)$ and radius 18
    d) with centre $(-2;0)$ and diameter 6
    e) with centre $(-5;-3)$ and radius $\sqrt{3}$
2. a) Find the equation of the circle with centre $(2;1)$ which passes through $(4;1)$.
    b) Where does it cut the line $y = x + 1$?
3. a) Find the equation of the circle with centre $(-3;-2)$ which passes through $(1;-4)$.
    b) Find the equation of the circle with centre $(3;1)$ which passes through $(2;5)$.
4. Find the centre and radius of the following circles:
    a) $(x+9)^2 + (y-6)^2 = 36$
    b) $\frac{1}{2}(x-2)^2 + \frac{1}{2}(y-9)^2 = 1$
    c) $(x+5)^2 + (y+7)^2 = 12$
    d) $x^2 + (y+4)^2 = 23$
    e) $3(x-2)^2 + 3(y+3)^2 = 12$
5. Find the x and y intercepts of the following graphs:
    a) $x^2 + (y-6)^2 = 100$
    b) $(x+4)^2 + y^2 = 16$
6. Find the centre and radius of the following circles:
    a) $x^2 + 6x + y^2 - 12y = -20$
    b) $x^2 + 4x + y^2 - 8y = 0$
    c) $x^2 + y^2 + 8y = 7$
    d) $x^2 - 6x + y^2 = 16$
    e) $x^2 - 5x + y^2 + 3y = -\frac{3}{4}$
    f) $x^2 - 6nx + y^2 + 10ny = 9n^2$
7. a) Find the gradient of the radius between the point $(4;5)$ on the circle and its centre $(-8;4)$.
    b) Find the gradient line tangent to the circle at the point $(4;5)$.
8. a) Given $(x-1)^2 + (y-7)^2 = 10$, determine the value(s) of x if $(x;4)$ lies on the circle.
    b) Find the gradient of the tangent to the circle at the point $(2;4)$.
9. Given a circle with the central coordinates $(a;b) = (-2;-2)$. Determine the equation of the tangent line of the circle at the point $(-1;3)$.
10. Find the equation of the tangent to the circle at point T.

11. $M(-2;-5)$ is a point on the circle $x^2 + y^2 + 18y + 61 = 0$. Determine the equation of the tangent at M.
12. $C(-4;2)$ is the centre of the circle passing through $(2;-3)$ and $Q(-10;p)$.

   a) Find the equation of the circle given.
   b) Determine the value of p.
   c) Determine the equation of the tangent to the circle at point Q.
13. Find the equation of the tangent to each circle:
   a) $x^2 + y^2 = 17$ at the point $(1;4)$
   b) $x^2 + y^2 = 25$ at the point $(3;4)$
   c) $(x+1)^2 + (y-2)^2 = 25$ at the point $(3;5)$
   d) $(x-2)^2 + (y-1)^2 = 13$ at the point $(5;3)$
14. Determine the equations of the tangents to the circle $x^2 + y^2 = 50$, given that both lines have an angle of inclination of $45°$.
15. The circle with centre $P(4;4)$ has a tangent AB at point B. The equation of AB is $y - x + 2 = 0$ and A lies on the y-axis.

   a) Determine the equation of PB.
   b) Determine the coordinates of B.
   c) Determine the equation of the circle.
   d) Describe in words how the circle must be shifted so that P is at the origin.

e) If the length of PB is tripled and the circle is shifted 2 units to the right and 1 unit up, determine the equation of the new circle.

f) The equation of a circle with centre A is $x^2 + y^2 + 5 = 16x + 8y - 30$ and the equation of a circle with centre B is $5x^2 + 5y^2 = 25$. Prove that the two circles touch each other.

16. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 296W 1b. 296X 1c. 296Y 1d. 296Z 1e. 2972 2. 2973
3. 2974 4a. 2975 4b. 2976 4c. 2977 4d. 2978 4e. 2979
5a. 297B 5b. 297C 6a. 297D 6b. 297F 6c. 297G 6d. 297H
6e. 297J 6f. 297K 7. 297M 8. 297N 9. 297P 10. 297Q
11. 297R 12. 297S 13a. 297T 13b. 297V 13c. 297W 13d. 297X
14. 297Y 15. 297Z

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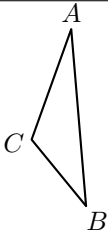
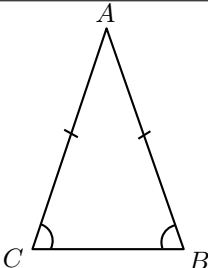
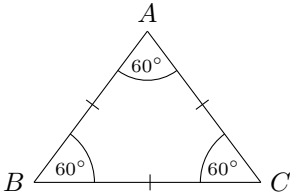
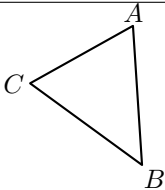
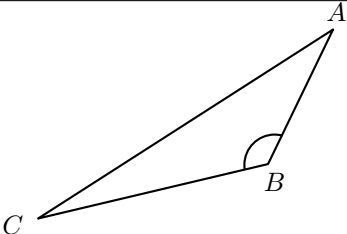
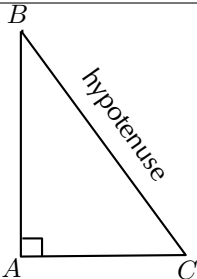

CHAPTER

Euclidean geometry

8.1 Revision EMCHY

Types of triangles EMCHZ

Name	Diagram	Properties
Scalene	A, C, B	All sides and angles are different.
Isosceles	A, C, B	Two sides are equal in length. The angles opposite the equal sides are also equal.
Equilateral	A, B, C, $60°$, $60°$, $60°$	All three sides are equal in length and all three angles are equal.
Acute-angled	A, C, B	Each of the three interior angles is less than $90°$.
Obtuse-angled	A, B, C	One interior angle is greater than $90°$.
Right-angled	B, hypotenuse, A, C	One interior angle is $90°$.

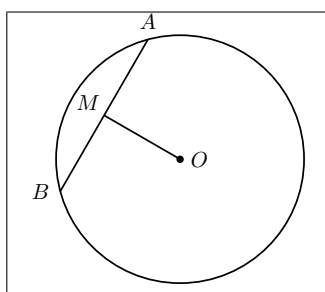
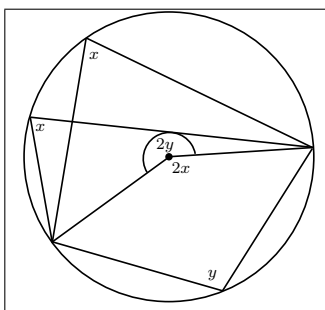
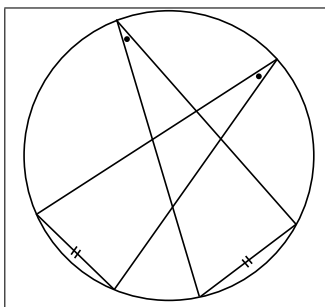
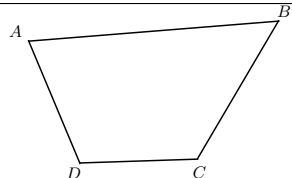
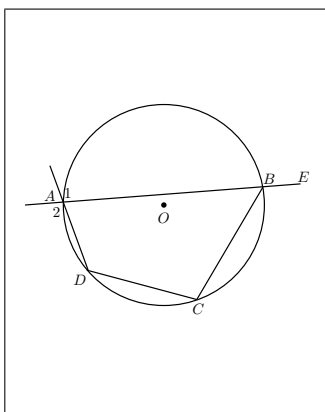
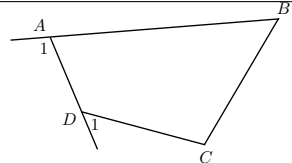
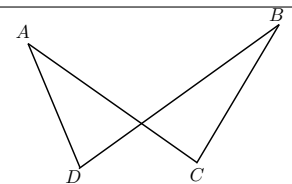
Congruent triangles

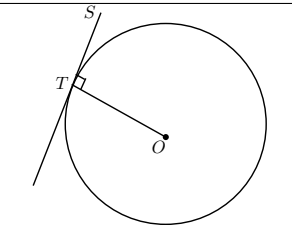
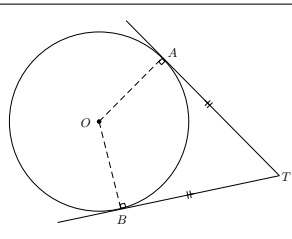
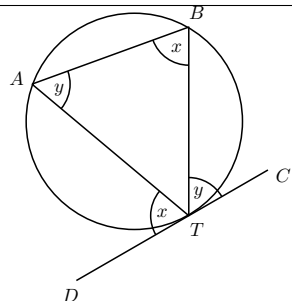
EMCJ2

Condition	Diagram
SSS (side, side, side)	$\triangle ABC \equiv \triangle EDF$
SAS (side, incl. angle, side)	$\triangle GHI \equiv \triangle JKL$
AAS (angle, angle, side)	$\triangle MNO \equiv \triangle PQR$
RHS (90°, hypotenuse, side)	$\triangle STU \equiv \triangle VWX$

Similar triangles

EMCJ3

Condition	Diagram
AAA (angle, angle, angle)	$\hat{A} = \hat{D},\ \hat{B} = \hat{E},\ \hat{C} = \hat{F}$ $\therefore \triangle ABC \mid\mid\mid \triangle DEF$
SSS (sides in prop.)	$\frac{MN}{RS} = \frac{ML}{RT} = \frac{NL}{ST}$ $\therefore \triangle MNL \mid\mid\mid \triangle RST$

	• If O is the centre and $OM \perp AB$, then $AM = MB$. • If O is the centre and $AM = MB$, then $A\hat{M}O = B\hat{M}O = 90°$. • If $AM = MB$ and $OM \perp AB$, then $\Rightarrow MO$ passes through centre O.
	If an arc subtends an angle at the centre of a circle and at the circumference, then the angle at the centre is twice the size of the angle at the circumference.
	Angles at the circumference subtended by arcs of equal length (or by the same arc) are equal.

Cyclic quadrilaterals EMCJ5

	If the four sides of a quadrilateral $ABCD$ are the chords of a circle with centre O, then: • $\hat{A} + \hat{C} = 180°$ Reason: (opp. ∠s cyclic quad. supp.) • $\hat{B} + \hat{D} = 180°$ Reason: (opp. ∠s cyclic quad. supp.) • $E\hat{B}C = \hat{D}$ Reason: (ext. ∠ cyclic quad. = int. opp ∠) • $\hat{A}_1 = \hat{A}_2 = \hat{C}$ Reason: (vert. opp. ∠s, ext. ∠ cyclic quad.)

Proving a quadrilateral is cyclic:

	If $\hat{A} + \hat{C} = 180°$ or $\hat{B} + \hat{D} = 180°$, then $ABCD$ is a cyclic quadrilateral.

	If $\hat{A}_1 = \hat{C}$ or $\hat{D}_1 = \hat{B}$, then $ABCD$ is a cyclic quadrilateral.

	If $\hat{A} = \hat{B}$ or $\hat{C} = \hat{D}$, then $ABCD$ is a cyclic quadrilateral.

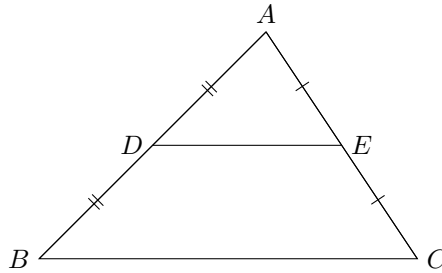
Tangents to a circle EMCJ6

	A tangent is perpendicular to the radius ($OT \perp ST$), drawn to the point of contact with the circle.

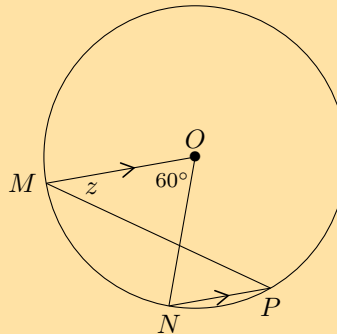
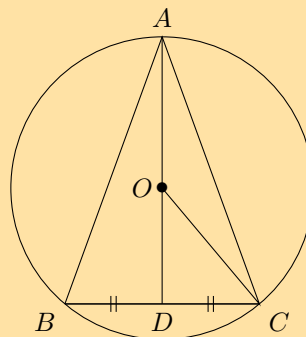
	If AT and BT are tangents to a circle with centre O, then: • $OA \perp AT$ (tangent $\perp$ radius) • $OB \perp BT$ (tangent $\perp$ radius) • $TA = TB$ (tangents from same point are equal)

	• If DC is a tangent, then $D\hat{T}A = T\hat{B}A$ and $C\hat{T}B = T\hat{A}B$. • If $D\hat{T}A = T\hat{B}A$ or $C\hat{T}B = T\hat{A}B$, then DC is a tangent touching at T.

The mid-point theorem

The line joining the mid-points of two sides of a triangle is parallel to the third side and equal to half the length of the third side.

Given: $AD = DB$ and $AE = EC$, we can conclude that $DE \parallel BC$ and $DE = \frac{1}{2}BC$.

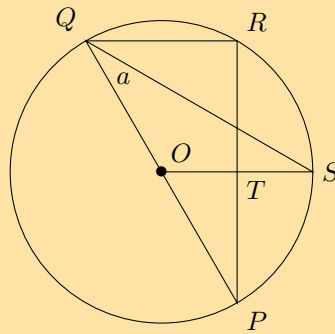
Exercise 8 – 1: Revision

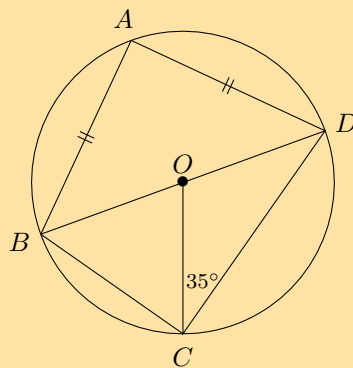
1. $MO \parallel NP$ in a circle with centre O. $M\hat{O}N = 60°$ and $O\hat{M}P = z$. Calculate the value of z, giving reasons.

2. O is the centre of the circle with $OC = 5$ cm and chord $BC = 8$ cm.

Determine the lengths of:

a) OD

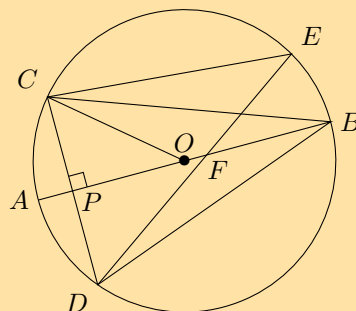
b) AD

c) AB

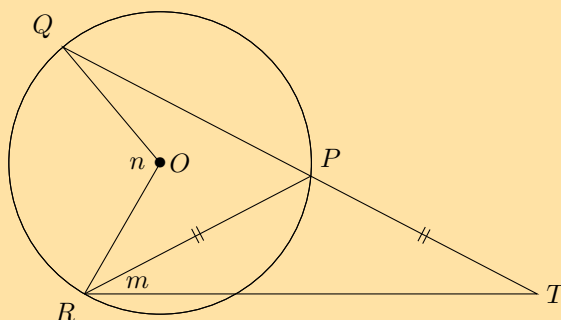
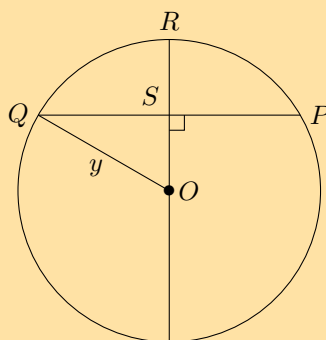
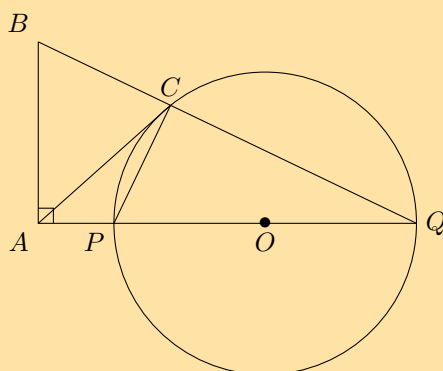
3. PQ is a diameter of the circle with centre O. SQ bisects $P\hat{Q}R$ and $P\hat{Q}S = a$.

a) Write down two other angles that are also equal to a.
b) Calculate $P\hat{O}S$ in terms of a, giving reasons.
c) Prove that OS is a perpendicular bisector of PR.

4. BD is a diameter of the circle with centre O. $AB = AD$ and $O\hat{C}D = 35°$.

Calculate the value of the following angles, giving reasons:

a) $O\hat{D}C$
b) $C\hat{O}D$
c) $C\hat{B}D$
d) $B\hat{A}D$
e) $A\hat{D}B$

5. O is the centre of the circle with diameter AB. $CD \perp AB$ at P and chord DE intersects AB at F.

Prove the following:

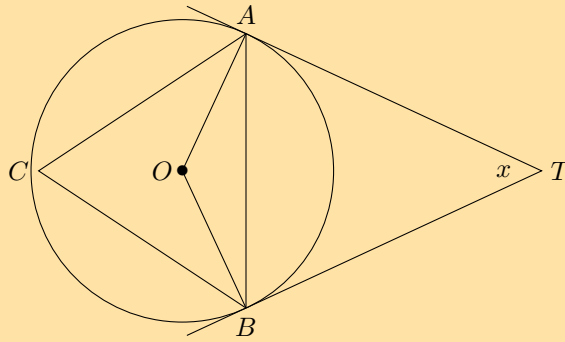
a) $C\hat{B}P = D\hat{B}P$
b) $C\hat{E}D = 2C\hat{B}A$
c) $A\hat{B}D = \frac{1}{2}C\hat{O}A$

6. QP in the circle with centre O is extended to T so that $PR = PT$. Express m in terms of n.

7. In the circle with centre O, $OR \perp QP$, $QP = 30$ mm and $RS = 9$ mm. Determine the length of y.

8. PQ is a diameter of the circle with centre O. QP is extended to A and AC is a tangent to the circle. $BA \perp AQ$ and BCQ is a straight line.

Prove the following:

a) $P\hat{C}Q = B\hat{A}P$

b) $BAPC$ is a cyclic quadrilateral

c) $AB = AC$

9. TA and TB are tangents to the circle with centre O. C is a point on the circumference and $A\hat{T}B = x$.

Express the following in terms of x, giving reasons:

a) $A\hat{B}T$ b) $O\hat{B}A$ c) $\hat{C}$

10. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 2982 2. 2983 3. 2984 4. 2985 5. 2986 6. 2987
7. 2988 8. 2989 9. 298B

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8.2 Ratio and proportion EMCJ8

Ratio

A ratio describes the relationship between two quantities which have the same units. We can use ratios to compare weights, heights, lengths, currencies, etc. A ratio is a comparison between two quantities of the same kind and has no units.

Example: if the length of a rectangle is 20 cm and the width is 60 cm, then we can express the ratio between the length and width of the rectangle as:

$$\begin{aligned}\text{length to width} &= 20 \text{ to } 60\\ &= 1 \text{ to } 3\end{aligned} \quad \text{Or} \quad \begin{aligned}\frac{\text{length}}{\text{width}} &= \frac{20}{60}\\ &= \frac{1}{3}\end{aligned} \quad \text{Or} \quad \begin{aligned}\text{length} : \text{width} &= 20 : 60\\ &= 1 : 3\end{aligned}$$

- The ratio of $\frac{1}{3}$ describes the length of the rectangle **relative** to its width.
- A ratio written as a fraction is usually given in its simplest form.
- A ratio gives no indication of actual length. For example,

$$\frac{\text{length}}{\text{width}} = \frac{50 \text{ cm}}{150 \text{ cm}} \text{ also gives a ratio of } \frac{1}{3}$$

$$\text{And } \frac{\text{length}}{\text{width}} = \frac{0{,}8 \text{ m}}{2{,}4 \text{ m}} \text{ also gives a ratio of } \frac{1}{3}$$

- Do not convert a ratio to a decimal (even though $\frac{1}{3}$ and $0{,}\dot{3}$ have the same numerical value).

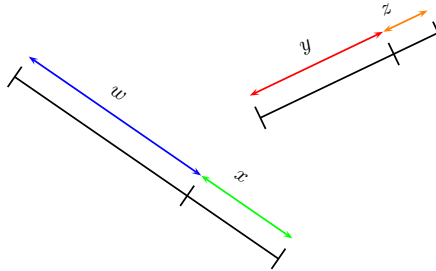
Proportion

Investigation: Predicting heights

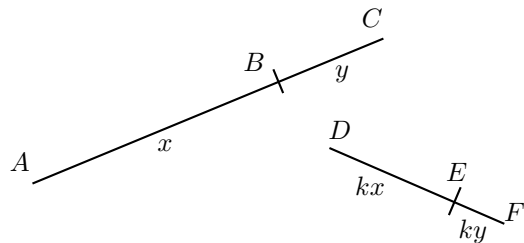
A record of heights is given.

If the ratio $\frac{\text{height of a person at two years old}}{\text{height of a person as an adult}}$ is 1 to 2, complete the table below:

Name	Height at two years	Height as an adult
Hendrik	84 cm	
Kagiso		162 cm
Linda	86 cm	
Mandisa	0,87 m	
Prashna		1 m 64 cm

Consider the diagram below:

If two or more ratios are equal to each other, then we say that they are in the same proportion. Proportionality describes the equality of ratios.

If $\frac{w}{x} = \frac{y}{z}$, then w and x are in the same proportion as y and z.

1. $wz = xy$
2. $\frac{x}{w} = \frac{z}{y}$
3. $\frac{w}{y} = \frac{x}{z}$
4. $\frac{y}{w} = \frac{z}{x}$

Given

$$\frac{AB}{BC} = \frac{x}{y} = \frac{kx}{ky} = \frac{DE}{EF}$$

The line segments AB and BC are in the same proportion as DE and EF. The following statements are also true:

Proportion	Reciprocal proportion	Cross multiplication
$\frac{AB}{BC} = \frac{DE}{FE}$	$\frac{BC}{AB} = \frac{FE}{DE}$	$AB\ .\ FE = BC\ .\ DE$
$\frac{AB}{AC} = \frac{DE}{DF}$	$\frac{AC}{AB} = \frac{DF}{DE}$	$AB\ .\ DF = AC\ .\ DE$
$\frac{BC}{AC} = \frac{EF}{DF}$	$\frac{AC}{BC} = \frac{DF}{EF}$	$BC\ .\ DF = AC\ .\ EF$

We can also substitute x, y, kx, and ky to show algebraically that the statements are true.

For example,

$$\begin{aligned} BC\ .\ DF &= y \times (kx + ky) \\ &= ky(x+y) \\ \text{And } AC\ .\ EF &= (x+y) \times ky \\ &= ky(x+y) \\ \therefore BC\ .\ DF &= AC\ .\ EF \end{aligned}$$

Exercise 8 – 2: Ratio and proportion

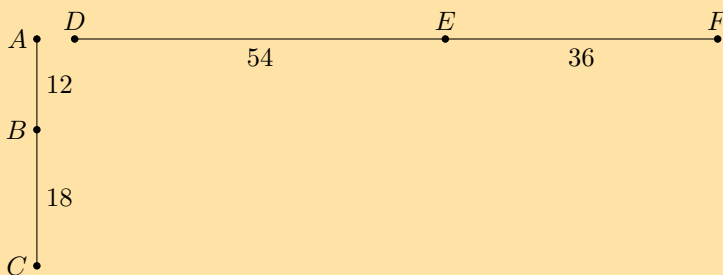
1. Solve for p:

   a) $\frac{8}{40} = \frac{p}{25}$

   b) $\frac{6}{9} = \frac{29+p}{54}$

   c) $\frac{3}{1+\frac{p}{4}} = \frac{4}{p+1}$

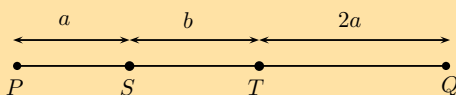
   d) $\frac{14}{100-p} = \frac{49}{343}$

2. A packet of 160 sweets contains red, blue and yellow sweets in the ratio of $3 : 2 : 3$ respectively. Determine how many sweets of each colour there are in the packet.
3. A mixture contains 2 parts of substance A for every 5 parts of substance B. If the total weight of the mixture is 50 kg, determine how much of substance B is in the mixture (correct to 2 decimal places).
4. Given the diagram below.

   Show that:

   a) $\frac{AB}{BC} = \frac{FE}{ED}$

   b) $\frac{AC}{BC} = \frac{FD}{EF}$

   c) $AB\ .\ DF = AC\ .\ FE$

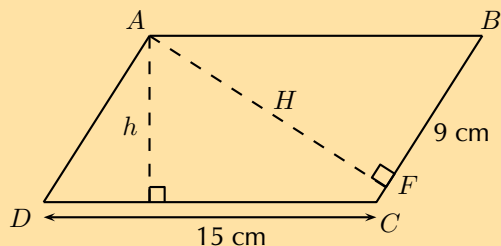
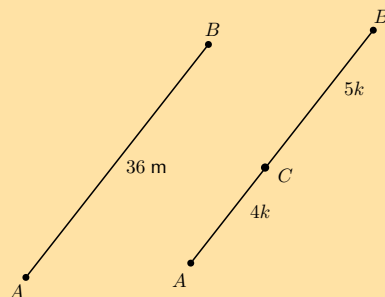
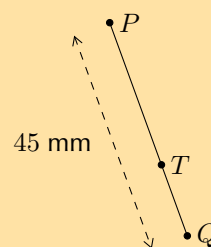
5. Consider the line segment shown below.

   Express the following in terms of a and b:

   a) $PT : ST$

   b) $\frac{PS}{TQ}$

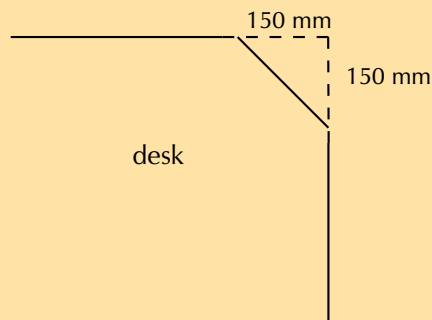
   c) $\frac{SQ}{PQ}$

   d) $QT : TS$

6. $ABCD$ is a parallelogram with $DC = 15$ cm, $h = 8$ cm and $BF = 9$ cm. Calculate the ratio $\frac{\text{area } ABF}{\text{area } ABCD}$.

7. $AB = 36$ m and C divides AB in the ratio $4 : 5$. Determine AC and CB.

8. If $PQ = 45$ mm and the ratio of $TQ : PQ$ is $2 : 3$, calculate PT and TQ.

9. Luke's biology notebook is 30 cm long and 20 cm wide. The dimensions of his desk are in the same proportion as the dimensions of his notebook.

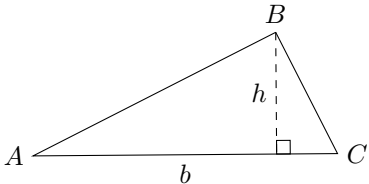
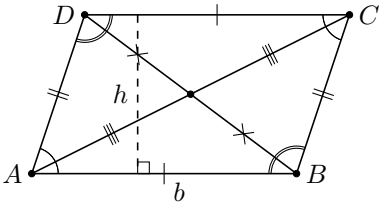
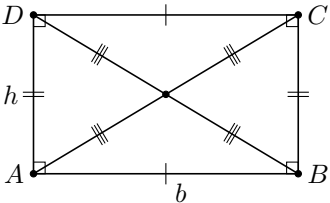
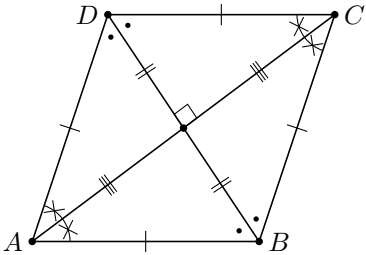
   a) If the desk is 90 cm wide, calculate the area of the top of the desk.

   b) Luke covers each corner of his desk with an isosceles triangle of cardboard, as shown in the diagram on the right.

      Calculate the new perimeter and area of the visible part of the top of his desk.

   c) Use this new area to calculate the dimensions of a square desk with the same desk top area.

10. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 298C 1b. 298D 1c. 298F 1d. 298G 2. 298H 3. 298J
4a. 298K 4b. 298M 4c. 298N 5a. 298P 5b. 298Q 5c. 298R
5d. 298S 6. 298T 7. 298V 8. 298W 9. 298X

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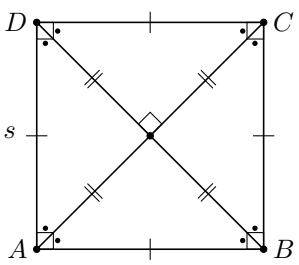
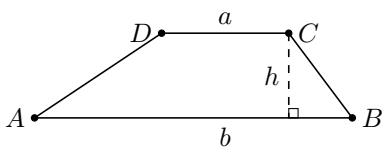
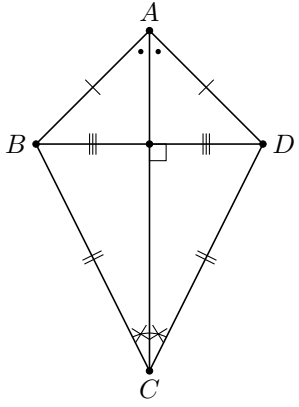
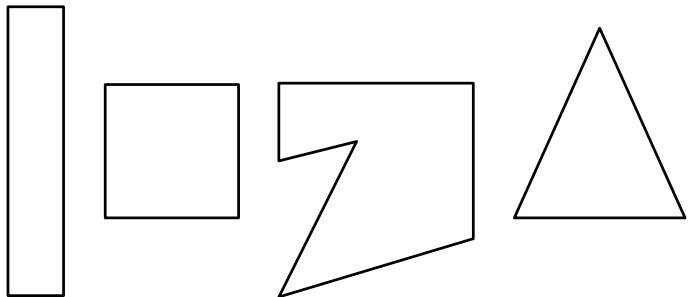

DEFINITION: *Proportionality in polygons*

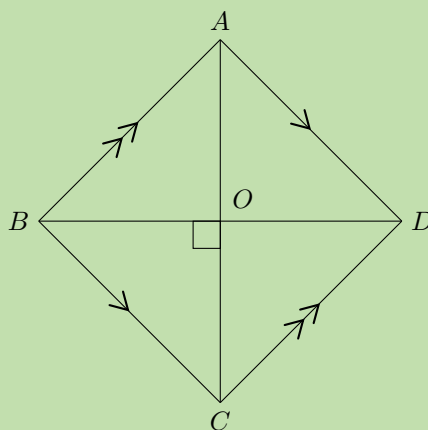
A plane, closed shape consisting of three or more line segments.

In previous grades we studied the properties of the following polygons:

Triangle		Area $= \frac{1}{2}b \times h$
Parallelogram		Area $= b \times h$
Rectangle		Area $= b \times h$
Rhombus		Area $= \frac{1}{2}AC \times BD$

Square		Area $= s^2$
Trapezium		Area $= \frac{1}{2}(a+b) \times h$
Kite		Area $= \frac{1}{2}(AC \times DB)$

Worked example 1: Properties of polygons

QUESTION

$ABCD$ is a rhombus with $BD = 12$ cm and $AB : BD = 3 : 4$.

Calculate the following (correct to two decimal places) and provide reasons:

1. Length of AB
2. Length of AO
3. Area of $ABCD$

SOLUTION

Step 1: Use the ratio to determine the length of AB

$$
\begin{aligned}
AB : BD &= 3 : 4 \\
\therefore \frac{AB}{BD} &= \frac{3}{4} \\
\frac{AB}{12} &= \frac{3}{4} \\
AB &= 12 \times \frac{3}{4} \\
&= 9 \text{ cm}
\end{aligned}
$$

Step 2: Calculate the length of AO

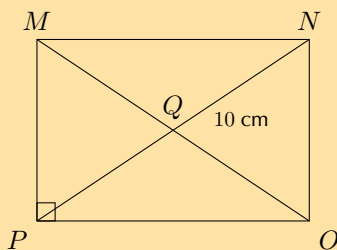
We use the properties of a rhombus and the theorem of Pythagoras to find AO.

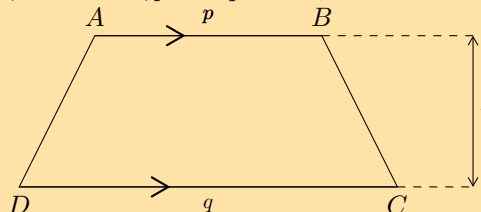
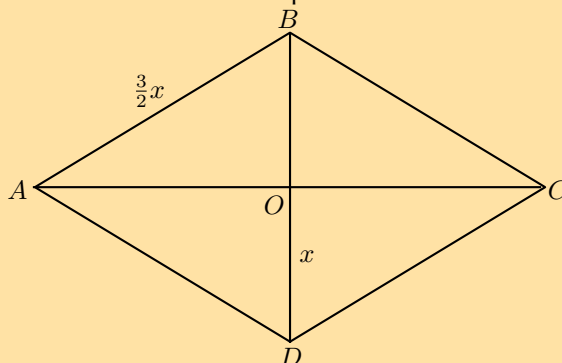
$$
\begin{aligned}
BD &= 12 \text{ cm} && \\
BO &= 6 \text{ cm} && \text{(diagonals bisect each other)} \\
\text{In } \triangle ABO, \quad A\hat{O}B &= 90^\circ && \text{(diagonals intersect at } \perp\text{)} \\
AO^2 &= AB^2 - BO^2 && \text{(Pythagoras)} \\
&= 9^2 - 6^2 && \\
\therefore AO &= \sqrt{45} && \\
&= 6{,}71 \text{ cm} &&
\end{aligned}
$$

Step 3: Determine the area of rhombus ABCD

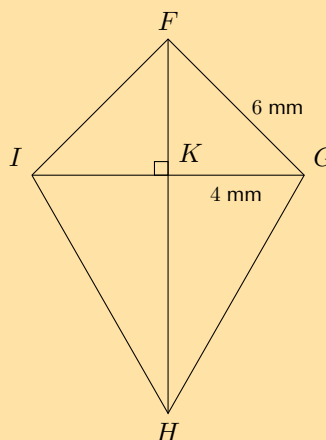
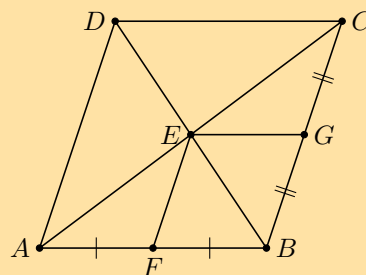
$$
\begin{aligned}
\text{Area } ABCD &= \frac{1}{2} AC \times BD \\
&= \frac{1}{2}(2 \times \sqrt{45})(12) \\
&= 80{,}50 \text{ cm}^2
\end{aligned}
$$

Exercise 8 – 3: Proportionality of polygons

1. $MNOP$ is a rectangle with $MN : NO = 5 : 3$ and $QN = 10$ cm.

   a) Calculate MN (correct to 2 decimal places).
   b) Calculate the area of $\triangle OPQ$ (correct to 2 decimal places).

2. Consider the trapezium $ABCD$ shown below. If $t : p : q = 2 : 3 : 5$ and area $ABCD = 288$ cm^2, calculate t, p and q.

3. $ABCD$ is a rhombus with sides of length $\frac{3}{2}x$ millimetres. The diagonals intersect at O and length $DO = x$ millimetres. Express the area of $ABCD$ in terms of x.

4. In the diagram below, $FGHI$ is a kite with $FG = 6$ mm, $GK = 4$ mm and $\frac{GH}{FI} = \frac{5}{2}$.

a) Determine FH (correct to the nearest integer).

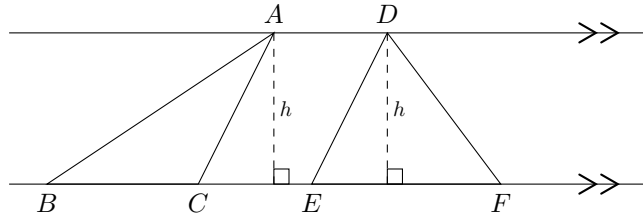
b) Calculate area $FGHI$.

5. $ABCD$ is a rhombus. F is the mid-point of AB and G is the mid-point of CB. Prove that $EFBG$ is also a rhombus.

6. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 298Y 2. 298Z 3. 2992 4. 2993 5. 2994

8.4 Triangles EMCJB

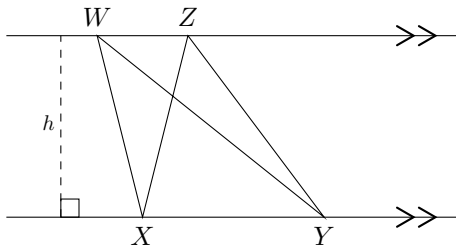
Proportionality of triangles EMCJC

1. In the diagram below, $\triangle ABC$ and $\triangle DEF$ have the same height (h) since both triangles are between the same parallel lines.

$$\text{Area } \triangle ABC = \frac{1}{2}BC \times h$$

$$\text{Area } \triangle DEF = \frac{1}{2}EF \times h \quad \text{and} \quad \frac{\text{Area } \triangle ABC}{\text{Area } \triangle DEF} = \frac{\frac{1}{2}BC \times h}{\frac{1}{2}EF \times h}$$

$$\text{Therefore } \frac{\text{Area } \triangle ABC}{\text{Area } \triangle DEF} = \frac{BC}{EF}$$

Triangles with equal heights have areas which are proportional to their bases.

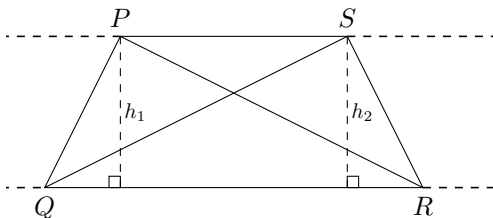
2. $\triangle WXY$ and $\triangle ZXY$ have the same base (XY) and the same height (h) since both triangles lie between the same parallel lines.

$$\text{Area } \triangle WXY = \frac{1}{2}XY \times h$$

$$\text{Area } \triangle ZXY = \frac{1}{2}XY \times h$$

$$\therefore \text{Area } \triangle WXY = \text{Area } \triangle ZXY$$

Triangles with equal bases and between the same parallel lines are equal in area.

3. $\triangle PQR$ and $\triangle SQR$ have the same base (QR) and are equal in area.

$$\text{Area } \triangle PQR = \text{Area } \triangle SQR$$

$$\frac{1}{2}QR \times h_1 = \frac{1}{2}QR \times h_2$$

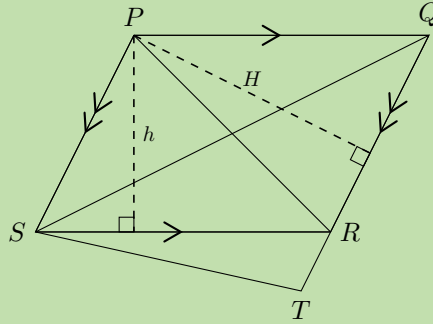
$$\therefore h_1 = h_2$$

$$\therefore PS \parallel QR$$

Triangles on the same side of the same base and equal in area, lie between parallel lines.

Worked example 2: Proportionality of triangles

QUESTION

Given parallelogram $PQRS$ with QR produced to T. $RS = 45$ cm, $QR = 30$ cm and $h = 10$ cm.

1. Calculate H.
2. If $TR : TQ = 1 : 4$, show that $\frac{\text{Area } \triangle STR}{\text{Area } \triangle PRQ} = \frac{1}{3}$.

SOLUTION

Step 1: Determine the length of H

We use the formula for area of a parallelogram to calculate H.

$$\begin{aligned}\text{Area } PQRS &= SR \times h \\ &= 45 \times 10 \\ &= 450 \text{ cm}^2 \\ \text{Area } PQRS &= QR \times H \\ 450 &= 30 \times H \\ \therefore H &= 15 \text{ cm}\end{aligned}$$

Step 2: Use proportionality to show that $\frac{\textbf{Area } \triangle \mathbf{STR}}{\textbf{Area } \triangle \mathbf{PRQ}} = \frac{\mathbf{1}}{\mathbf{3}}$

We are given the ratio $TR : TQ = 1 : 4$.

$$\begin{aligned}\frac{TR}{TQ} &= \frac{1}{4} \\ \text{So then } \frac{RQ}{TQ} &= \frac{3}{4} \\ \text{And } \frac{TR}{RQ} &= \frac{TR}{TQ} \times \frac{TQ}{RQ} \\ &= \frac{1}{4} \times \frac{4}{3} \\ &= \frac{1}{3}\end{aligned}$$

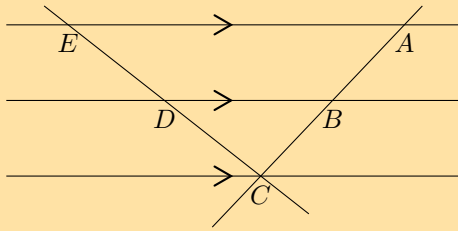
$$\begin{aligned}\text{Area } \triangle STR &= \frac{1}{2} TR \times H \quad (PS \parallel QT, \text{ equal heights}) \\ \text{Area } \triangle PRQ &= \frac{1}{2} RQ \times H\end{aligned}$$

$$\begin{aligned}\therefore \frac{\text{Area } \triangle STR}{\text{Area } \triangle PRQ} &= \frac{\frac{1}{2} TR \times H}{\frac{1}{2} RQ \times H} \\ &= \frac{TR}{RQ} \\ &= \frac{1}{3}\end{aligned}$$

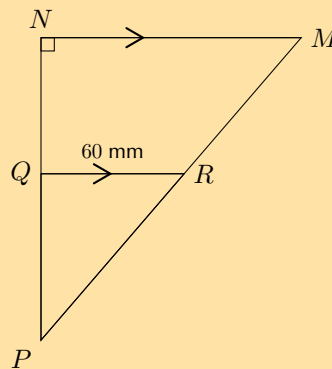
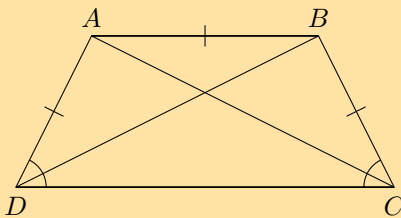
Exercise 8 – 4: Proportionality of triangles

1. The diagram below shows three parallel lines cut by two transversals EC and AC such that $ED : DC = 4 : 6$.

Determine:

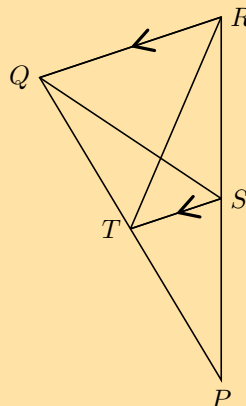
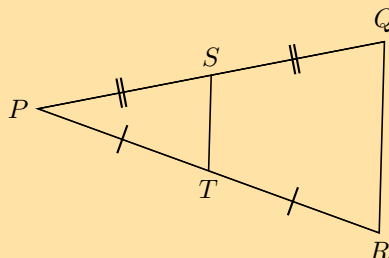
a) $\frac{BC}{AB}$

b) $AB : AC$

c) The lengths of AC and ED, if it is given that $AB = 12$ mm.

2. In right-angled $\triangle MNP$, QR is drawn parallel to NM, with R the mid-point of MP. $NP = 16$ cm and $RQ = 60$ mm. Determine QP and RP.

3. Given trapezium $ABCD$ with $DA = AB = BC$ and $A\hat{D}C = B\hat{C}D$.

a) Prove that BD bisects $\hat{D}$.

b) Prove that the two diagonals are equal in length.

c) If $DC : AB = 5 : 4$, show that area $ABCD = 2{,}25 \times$ area $\triangle ABC$.

4. In the diagram to the right, $\triangle PQR$ is given with $QR \parallel TS$. Show that area $\triangle PQS =$ area $\triangle PRT$.

5. In Grade 10 we proved the mid-point theorem using congruent triangles.
   a) Complete the following statement of the mid-point theorem:
   "The line joining ...... of a triangle is ...... to the third side and equal to ......"
   b) In $\triangle PQR$, T and S are the mid-points of PR and PQ respectively. Prove $TS \parallel RQ$.

   c) Write down the converse of the mid-point theorem.
6. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 2995 2. 2996 3. 2997 4. 2998 5. 2999

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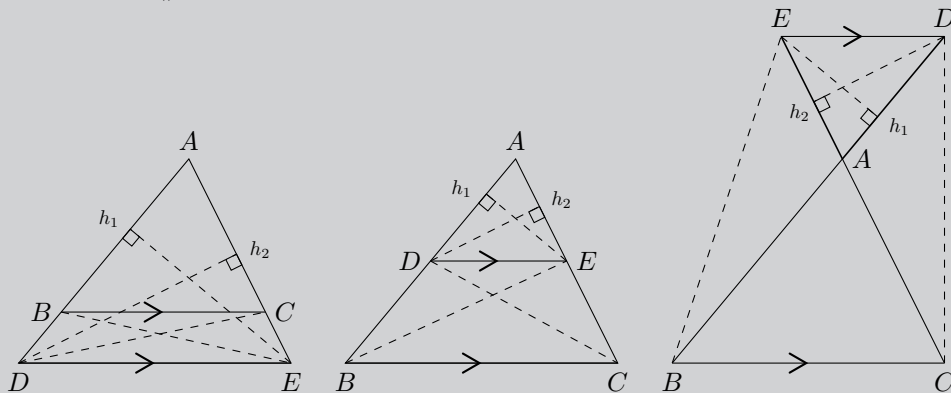
Theorem: Proportion theorem

STATEMENT

A line drawn parallel to one side of a triangle divides the other two sides of the triangle in the same proportion.

(Reason: line $\parallel$ to one side $\triangle$)

Given:

$\triangle ABC$ with line $DE \parallel BC$

Required to prove:

$$\frac{AD}{DB} = \frac{AE}{EC}$$

PROOF

Draw h_1 from E perpendicular to AD, and h_2 from D perpendicular to AE.

Draw BE and CD.

$$\frac{\text{Area }\triangle ADE}{\text{Area }\triangle BDE} = \frac{\frac{1}{2}AD.h_1}{\frac{1}{2}DB.h_1} = \frac{AD}{DB}$$

$$\frac{\text{Area }\triangle ADE}{\text{Area }\triangle CED} = \frac{\frac{1}{2}AE.h_2}{\frac{1}{2}EC.h_2} = \frac{AE}{EC}$$

$$\text{but Area }\triangle BDE = \text{Area }\triangle CED \quad \text{(equal base and height)}$$

$$\therefore \frac{\text{Area }\triangle ADE}{\text{Area }\triangle BDE} = \frac{\text{Area }\triangle ADE}{\text{Area }\triangle CED}$$

$$\therefore \frac{AD}{DB} = \frac{AE}{EC}$$

Similarly, we use the same method to show that:

$$\frac{AD}{AB} = \frac{AE}{AC} \quad \text{and} \quad \frac{AB}{BD} = \frac{AC}{CE}$$

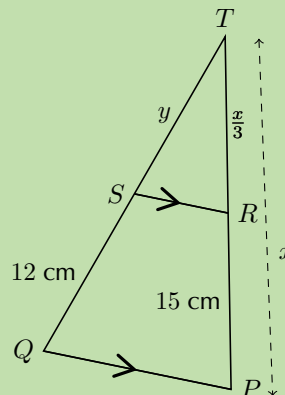
Converse: proportion theorem

If a line divides two sides of a triangle in the same proportion, then the line is parallel to the third side.

(Reason: line divides sides in prop.)

Worked example 3: Proportion theorem

QUESTION

In $\triangle TQP$, $SR \parallel QP$, $SQ = 12$ cm and $RP = 15$ cm. If $TR = \frac{x}{3}$, $TP = x$ and $TS = y$, determine the values of x and y, giving reasons.

SOLUTION

Step 1: Use the proportion theorem to determine the values of x and y

Consider TP:

$$\begin{aligned} RP &= x - \tfrac{1}{3}x \\ &= \tfrac{2}{3}x \end{aligned}$$

$$\begin{aligned} \therefore 15 &= \tfrac{2}{3}x \\ x &= 22{,}5 \text{ cm} \end{aligned}$$

$$\begin{aligned} TR &= \tfrac{1}{3}(22{,}5) \\ &= 7{,}5 \text{ cm} \end{aligned}$$

Consider TQ:

$$\frac{TS}{SQ} = \frac{TR}{RP} \qquad \text{(line } \parallel \text{ one side of } \triangle\text{)}$$

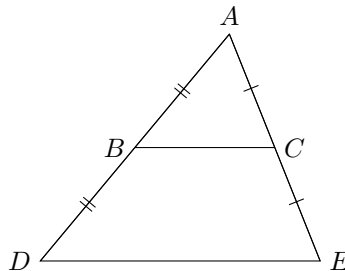
$$\frac{y}{12} = \frac{7{,}5}{15}$$

$$\therefore y = 6 \text{ cm}$$

Special case: the mid-point theorem

A corollary of the proportion theorem is the mid-point theorem: the line joining the mid-points of two sides of a triangle is parallel to the third side and equal to half the length of the third side.

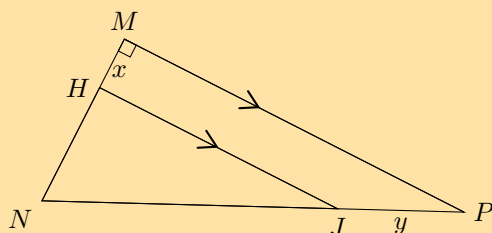
If $AB = BD$ and $AC = CE$, then $BC \parallel DE$ and $BC = \frac{1}{2}DE$.

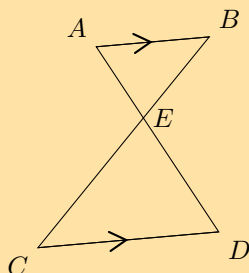
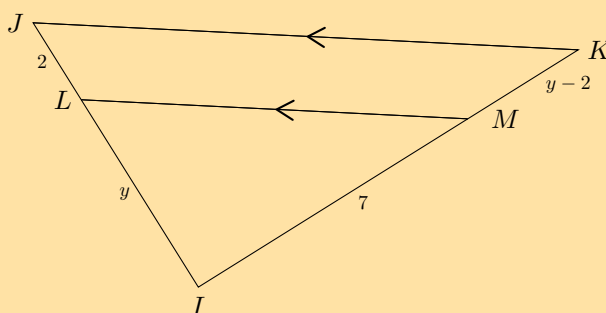
We also know that $\frac{AC}{CE} = \frac{AB}{BD}$.

Converse: the mid-point theorem

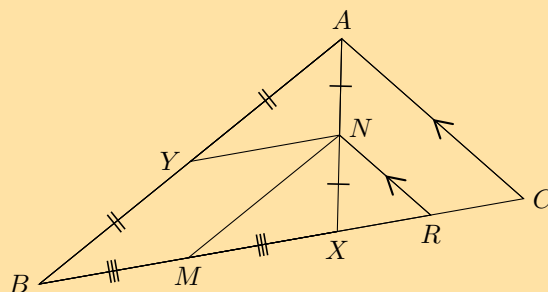
The line drawn from the mid-point of one side of a triangle parallel to another side, bisects the third side of the triangle.

If $AB = BD$ and $BC \parallel DE$, then $AC = CE$.

Exercise 8 – 5: Proportion theorem

1. In $\triangle MNP$, $\hat{M} = 90°$ and $HJ \parallel MP$.
   $HN : MH = 3 : 1$, $HM = x$ and $JP = y$.

   a) Calculate $JP : NP$.
   b) Calculate $\frac{\text{area } \triangle HNJ}{\text{area } \triangle MNP}$.

2. Use the given diagram to prove the Proportion Theorem.

3. In the diagram below, $JL = 2, LI = y, IM = 7$ and $MK = y - 2$.
   If $LM \parallel JK$, calculate y (correct to two decimal places).

4. Write down the converse of the proportion theorem and illustrate with a diagram.

5. In $\triangle ABC$, X is a point on BC. N is the mid-point of AX, Y is the mid-point of AB and M is the mid-point of BX.

   a) Prove that $YBMN$ is a parallelogram.
   b) Prove that $MR = \frac{1}{2}BC$.

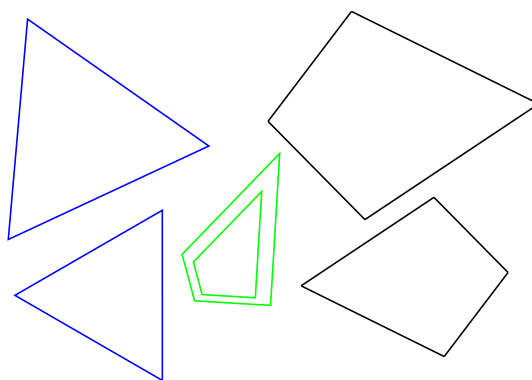
6. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

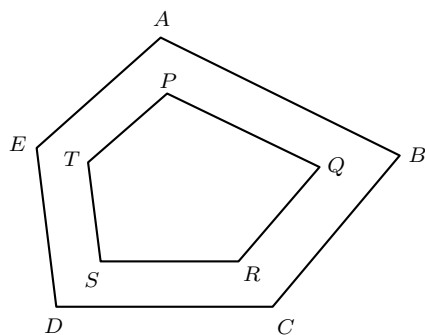
Check answers online with the exercise code below or click on 'show me the answer'.

1. 299B 2. 299C 3. 299D 4. 299F 5. 299G

8.5 Similarity EMCJD

Similar polygons EMCJF

Polygons are similar if they are the same shape but differ in size. In other words, one polygon is an enlargement of the other. Congruent polygons are also similar since they are the same shape and the same size, only their position or orientation is different.

Two polygons with the same number of sides are similar when:

1. All pairs of corresponding angles are equal, **and**
2. All pairs of corresponding sides are in the same proportion.

In the diagram above, polygon $ABCDE$ is similar to polygon $PQRST$ if:

Condition 1:

$$\hat{A} = \hat{P}, \quad \hat{B} = \hat{Q}, \quad \hat{C} = \hat{R}, \quad \hat{D} = \hat{S}, \quad \hat{E} = \hat{T}$$

AND

Condition 2:

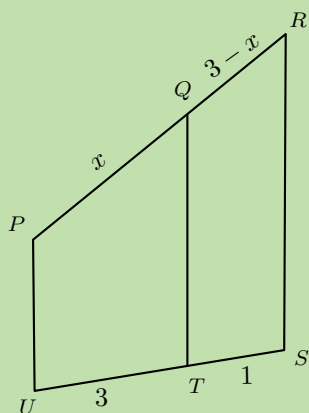
$$\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CD}{RS} = \frac{DE}{ST} = \frac{EA}{TP}$$

Important:

- Both conditions must be true for two polygons to be similar.
- If we are given two polygons that are similar, then we know that both conditions are true.

Worked example 4: Similar polygons

QUESTION

Polygons $PQTU$ and $PRSU$ are similar. Determine the value of x.

SOLUTION

Step 1: Identify pairs of corresponding sides

Since the two polygons are similar,

$$\begin{aligned} \frac{PQ}{PR} &= \frac{TU}{SU} \\ \therefore \frac{x}{x+(3-x)} &= \frac{3}{3+1} \\ \therefore \frac{x}{3} &= \frac{3}{4} \\ \therefore x &= \frac{9}{4} \end{aligned}$$

Exercise 8 – 6: Similar polygons

1. Determine whether or not the following polygons are similar, giving reasons.

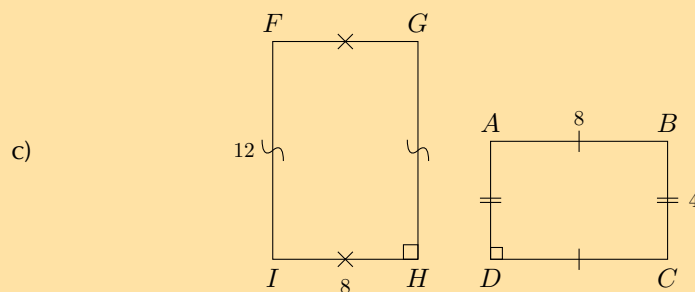
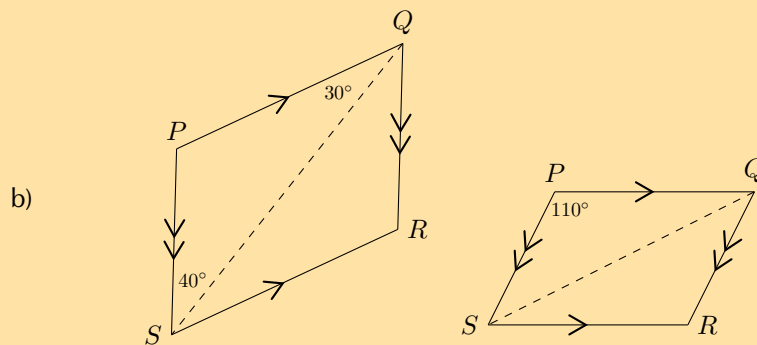
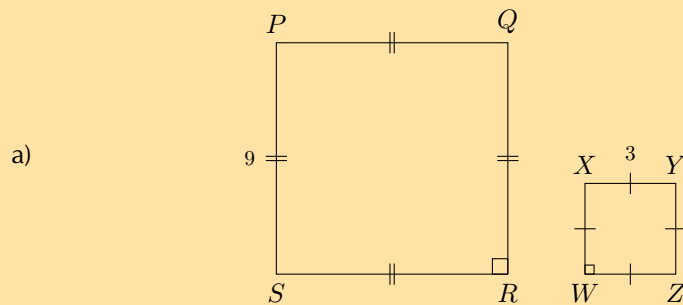
a)

P Q S R 9 X Y W Z 3

b)

P Q R S 30° 40° P Q R S 110°

c)

F G I H 12 8 A B D C 8 4

2. Are the following statements true or false? If false, state reasons or draw an appropriate diagram.
    a) All squares are similar.
    b) All rectangles are similar.
    c) All rhombi are similar.
    d) All congruent polygons are similar.
    e) All similar polygons are congruent.
    f) All congruent triangles are similar.
    g) Isosceles triangles are similar.
    h) Equilateral triangles are similar.
3. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 299H 1b. 299J 1c. 299K 2a. 299M 2b. 299N 2c. 299P
2d. 299Q 2e. 299R 2f. 299S 2g. 299T 2h. 299V

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Similarity of triangles EMCJG

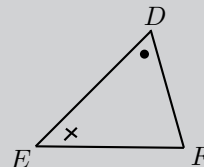
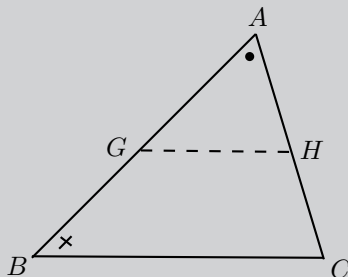
To prove two polygons are similar, we need to show that two conditions are true: (a) all pairs of corresponding angles are equal and (b) all pairs of corresponding sides are in the same proportion.

To prove two triangles are similar, we need only show that **one** of the conditions is true. If one of the conditions is true for two triangles, then the other condition is also true.

Theorem: Equiangular triangles are similar

STATEMENT

Equiangular triangles are similar.

(Reason: equiangular $\triangle$s)

Given:
$\triangle ABC$ and $\triangle DEF$
with $\hat{A} = \hat{D}$; $\hat{B} = \hat{E}$; $\hat{C} = \hat{F}$

Required to prove:
$$\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$$

PROOF

Construction: mark G on AB so that $AG = DE$, and mark H on AC so that $AH = DF$.

In $\triangle AGH$ and $\triangle DEF$

$$
\begin{aligned}
AG &= DE && \text{(by construction)}\\
AH &= DF && \text{(by construction)}\\
\hat{A} &= \hat{D} && \text{(given)}\\
\triangle AGH &\equiv \triangle DEF && \text{(SAS)}\\
\therefore A\hat{G}H &= \hat{E} &&\\
\hat{E} &= \hat{B} && \text{(given)}\\
\therefore A\hat{G}H &= \hat{B} &&\\
\therefore GH &\parallel BC && \text{(corresp. } \angle \text{s equal)}
\end{aligned}
$$

$$
\begin{aligned}
\frac{AB}{AG} &= \frac{AC}{AH} && \text{(prop. theorem)}\\
\therefore \frac{AB}{DE} &= \frac{AC}{DF} && (AG = DE,\ AH = DF)
\end{aligned}
$$

Similarly, by constructing Q on CA so that $CQ = FD$, and mark P on BC so that $CP = FE$.

$$\frac{CA}{FD} = \frac{CB}{FE}$$

$$\therefore \frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$$

$$\therefore \triangle ABC \ ||| \ \triangle DEF$$

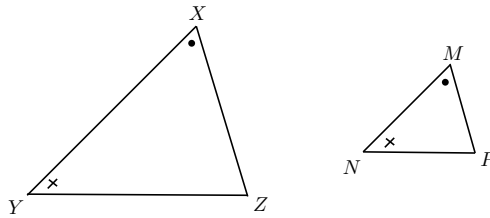
Therefore equiangular triangles are similar.

Notation

- The symbol for congruency is $\equiv$.
- The symbol for similarity is $|||$.
- Be careful to label similar triangles correctly.
  For example, if $\hat{P} = \hat{B}$, $\hat{Q} = \hat{A}$ and $\hat{R} = \hat{C}$, then $\triangle PQR \ ||| \ \triangle BAC$.
  Do not write $\triangle PQR \ ||| \ \triangle ABC$.
- $\triangle PQR \ ||| \ \triangle BAC$ also indicates which sides of the triangles are in the same proportion:

$$\frac{PQ}{BA} = \frac{QR}{AC} = \frac{PR}{BC}$$

Proving equiangular triangles are similar:

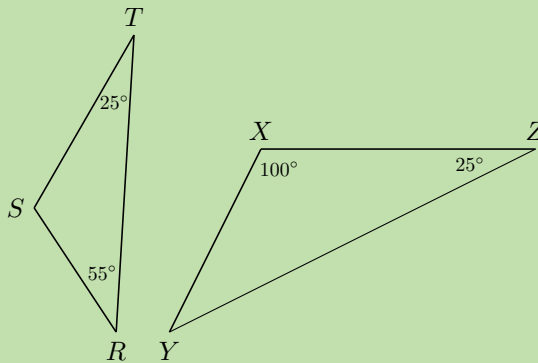
The sum of the interior angles of any triangle is $180°$. If we know that two pairs of angles are equal, then the remaining angle in each triangle must also be equal. Therefore the two triangles are similar.

$$\hat{X} = 180° - \left(\hat{Y} + \hat{Z}\right) \quad \text{(sum of int. } \angle\text{s of } \triangle)$$
$$\hat{M} = 180° - \left(\hat{N} + \hat{P}\right) \quad \text{(sum of int. } \angle\text{s of } \triangle)$$
$$\therefore \hat{X} = \hat{M}$$

$$\therefore \triangle XYZ \quad ||| \quad \triangle MNP \quad \text{(equiangular)}$$

In other words, to prove that two triangles are equiangular, we need only show that **two** pairs of angles are equal.

Worked example 5: Similarity of triangles

QUESTION

Prove that $\triangle XYZ$ is similar to $\triangle SRT$.

SOLUTION

Step 1: Calculate the unknown angles in each triangle

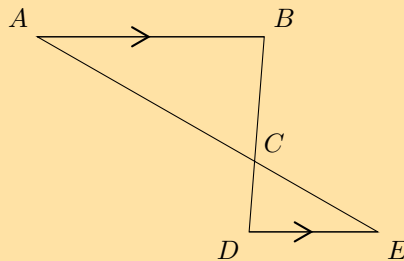
In $\triangle XYZ$: $\hat{Y} = 180° - (100° + 25°)$ (sum of int. $\angle$s of $\triangle$)
$= 55°$

In $\triangle SRT$: $\hat{S} = 180° - (55° + 25°)$ (sum of int. $\angle$s of $\triangle$)
$= 100°$

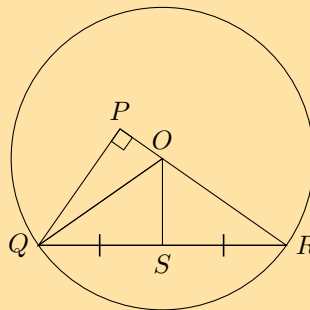
Step 2: Prove that the two triangles are similar

$$\begin{aligned} &\text{In } \triangle XYZ \text{ and } \triangle SRT: \\ &\hat{X} = \hat{S} = 100° && \text{(proved)} \\ &\hat{Z} = \hat{T} = 25° && \text{(given)} \\ &\therefore \triangle XYZ \ \ ||| \ \ \triangle SRT && \text{(AAA)} \end{aligned}$$

Exercise 8 – 7: Similarity of triangles

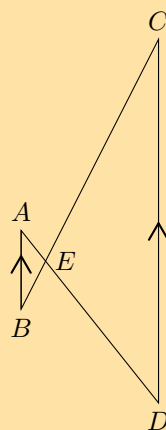
1. In the diagram below, $AB \parallel DE$.

a) Prove $\triangle ABC \ ||| \ \triangle EDC$.

b) If $\frac{AC}{AE} = \frac{5}{7}$ and $AB = 4$ cm, calculate the length of DE (correct to one decimal place).

2. In circle O, $RP \perp PQ$.

a) Prove $\triangle PRQ \ ||| \ \triangle SRO$.

b) Prove $\frac{OR}{SR} = \frac{QR}{PR}$.

c) If $SR = 18$ mm and $QP = 20$ mm, calculate the radius of circle O (correct to one decimal place).

3. Given the opposite figure with the following lengths, find AE, EC and BE. $BC = 15$ cm, $AB = 4$ cm, $CD = 18$ cm and $ED = 9$ cm.

4. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 299W 2. 299X 3. 299Y

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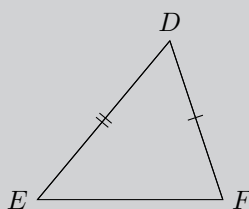
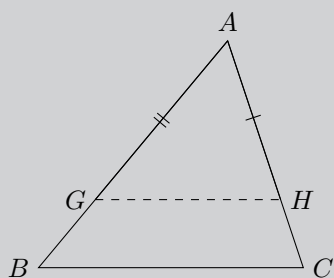
Theorem: Triangles with sides in proportion are similar

STATEMENT

If the corresponding sides of two triangles are in proportion, then the two triangles are similar.

(Reason: sides of $\triangle$s in prop.)

Given:

$\triangle ABC$ and $\triangle DEF$ with $\dfrac{AB}{DE} = \dfrac{AC}{DF} = \dfrac{BC}{EF}$

Required to prove:

$\hat{A} = \hat{D}$, $\hat{B} = \hat{E}$ and $\hat{C} = \hat{F}$

PROOF

Construction: draw GH such that $AG = DE$ and $AH = DF$.

$$\frac{AB}{DE} = \frac{AC}{DF} \qquad \text{(given)}$$

$$\frac{AB}{AG} = \frac{AC}{AH} \qquad \text{(by construction)}$$

$$\therefore GH \parallel BC \qquad \text{(sides in prop.)}$$

$$\therefore \hat{B} = A\hat{G}H \qquad \text{(corresp. } \angle\text{s, } GH \parallel BC\text{)}$$

$$\text{and } \hat{C} = A\hat{H}G \qquad \text{(corresp. } \angle\text{s, } GH \parallel BC\text{)}$$

$$\therefore \triangle AGH \ ||| \ \triangle ABC \qquad \text{(equiangular } \triangle\text{s)}$$

$$\therefore \frac{GH}{BC} = \frac{AG}{AB}$$

$$= \frac{DE}{AB} \qquad (AG = DE\text{, by construction})$$

$$= \frac{EF}{BC} \qquad \left(\text{given } \frac{AB}{DE} = \frac{BC}{EF}\right)$$

$$\therefore GH = EF$$

$$\therefore \triangle AGH \equiv \triangle DEF \qquad \text{(SSS)}$$

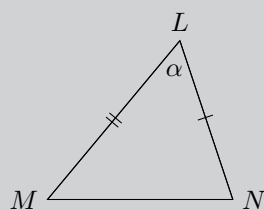
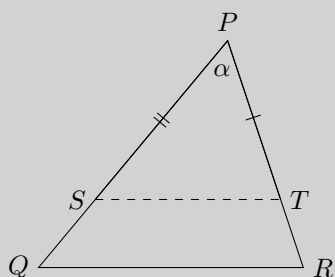
$\therefore \triangle ABC$ and $\triangle DEF$ are equiangular

$$\therefore \triangle ABC \ ||| \ \triangle DEF$$

Theorem: Triangles with two sides in proportion and equal included angles, are similar (PROOF NOT FOR EXAMS)

STATEMENT

If two sides of one triangle are in proportion to two sides of another triangle and the included angles are equal, then the two triangles are similar.

(Reason: $\triangle$s with 2 sides in prop. and equal incl. $\angle$s)

Given:

$\triangle PQR$ and $\triangle LMN$ with $\frac{LM}{PQ} = \frac{LN}{PR}$ and $\hat{L} = \hat{P} = \alpha$

Required to prove:

$\hat{M} = \hat{Q}$ and $\hat{N} = \hat{R}$

PROOF

Construction: draw ST such that $PS = LM$ and $PT = LN$.

In $\triangle PQR$ and $\triangle LMN$:

$PS = LM$ (by construction)

$\hat{P} = \hat{L}$ (given)

$PT = LN$ (by construction)

$\therefore \triangle PST \equiv \triangle LMN$ (SAS)

$\therefore P\hat{S}T = \hat{M}$

$P\hat{T}S = \hat{N}$

and $\frac{PS}{PQ} = \frac{PT}{PR}$ (by construction)

$\therefore ST \parallel QR$ (sides in prop.)

$\therefore \hat{Q} = P\hat{S}T = \hat{M}$ (corresp. $\angle$s, $ST \parallel QR$)

and $\hat{R} = P\hat{T}S = \hat{N}$ (corresp. $\angle$s, $ST \parallel QR$)

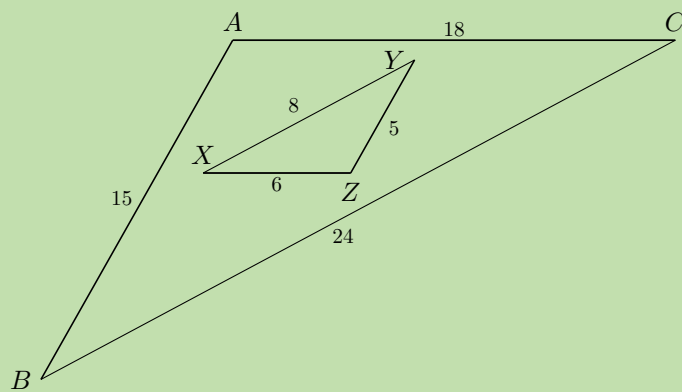
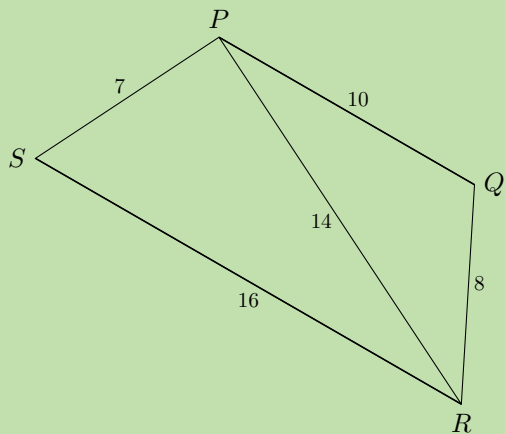
$\therefore \triangle PQR \ ||| \ \triangle LMN$ (equiangular $\triangle$s)

Proving triangles with sides in proportion are similar:

To prove that the sides of two triangles are in the same proportion, we must show that all **three** pairs of corresponding sides are in the same proportion.

Worked example 6: Similarity of triangles

QUESTION

1. In the diagram below, is $\triangle ABC \ ||| \ \triangle ZYX$?
   Show calculations.

2. Consider the diagram given below. Determine whether or not $\triangle PQR \ ||| \ \triangle SPR$.
   Show calculations.

(Diagrams are not drawn to scale)

SOLUTION

Step 1: Investigate proportionality of $\triangle ABC$ and $\triangle ZYX$

For similar triangles, the order in which the triangles are labelled indicates which pairs of sides correspond.

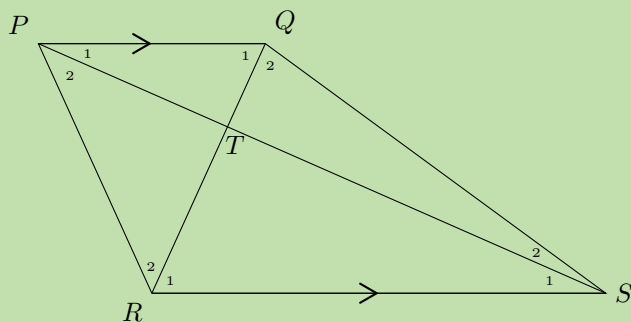
$$\begin{aligned}
&\text{In } \triangle ABC \text{ and } \triangle ZYX: \\
&\frac{AB}{ZY} = \frac{15}{5} = \frac{3}{1} \\
&\frac{AC}{ZX} = \frac{18}{6} = \frac{3}{1} \\
&\frac{BC}{YX} = \frac{24}{8} = \frac{3}{1} \\
&\therefore \triangle ABC \ ||| \ \triangle ZYX \qquad \text{(SSS)}
\end{aligned}$$

Step 2: Investigate proportionality of $\triangle PQR$ and $\triangle RPS$

$$\begin{aligned}
&\text{In } \triangle PQR \text{ and } \triangle RPS: \\
&\frac{PQ}{RP} = \frac{10}{14} = \frac{5}{7} \\
&\frac{PR}{RS} = \frac{14}{16} = \frac{7}{8} \\
&\frac{QR}{PS} = \frac{8}{7} \\
&\therefore \triangle PQR \text{ is not similar to } \triangle RPS
\end{aligned}$$

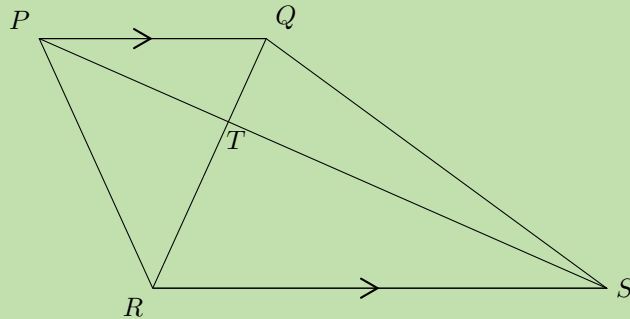
Worked example 7: Similarity of triangles

QUESTION

$PQSR$ is a trapezium, with $PQ \parallel RS$. Prove that $PT \cdot RT = ST \cdot QT$.

SOLUTION

Step 1: Identify triangles

We are required to prove that $PT\ .\ RT = ST\ .\ QT$, which we can also write as the ratio $\frac{PT}{QT} = \frac{ST}{RT}$.

To determine which two triangles we must consider, examine the lengths given in the ratio.

Step 2: Prove triangles are equiangular

In $\triangle PTQ$ and $\triangle STR$: $\hat{P}_1 = \hat{S}_1$ (alt. $\angle$s, $PQ \parallel RS$)

$\hat{Q}_1 = \hat{R}_1$ (alt. $\angle$s, $PQ \parallel RS$)

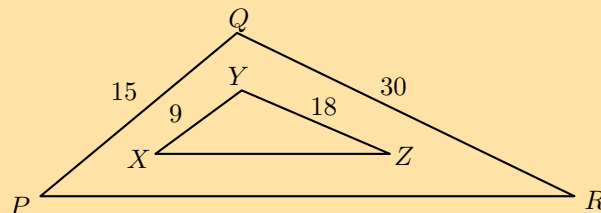
$\therefore \triangle PTQ \quad ||| \quad \triangle STR$ (AAA)

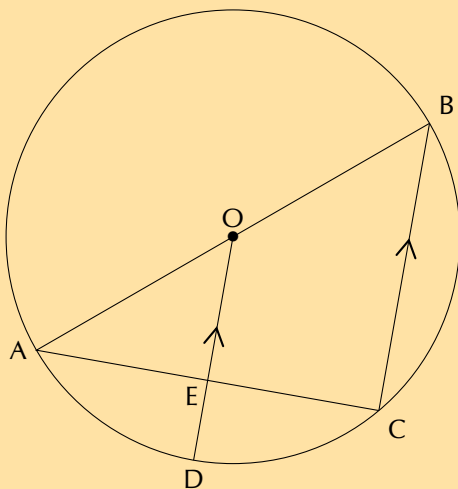
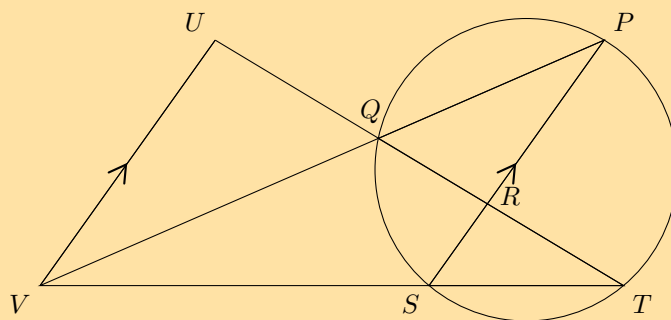
Step 3: Use proportionality

$$\frac{PT}{TQ} = \frac{ST}{TR} \qquad (\triangle PTQ \quad ||| \quad \triangle STR)$$

$$\therefore PT\ .\ TR = ST\ .\ TQ$$

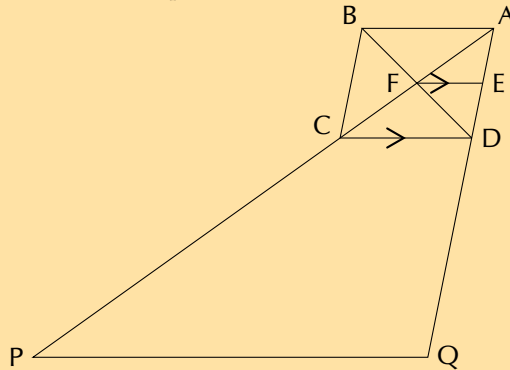
Exercise 8 – 8: Similarity of triangles

1. Consider the diagram given below. $PR = 20$ units and $XZ = 12$ units. Is $\triangle XYZ \quad ||| \quad \triangle PQR$? Give reasons.

2. AB is a diameter of the circle $ABCD$. OD is drawn parallel to BC and meets AC in E.

   If the radius is 10 cm and $AC = 16$ cm, calculate the length of ED.
   [NCS, Paper 3, November 2011]

3. P, Q, S and T are on the circumference of the circle.
   TS is produced to V so that $SV = 2TS$.
   TRQ is produced to U so that $VU \parallel SRP$.

   Prove, with reasons, that:

   a) $\dfrac{TR}{RU} = \dfrac{1}{3}$
   b) $\triangle TQV \; ||| \; \triangle PSV$
   c) $QV \, . \, PV = 6TS^2$
   d) $\triangle UQV \; ||| \; \triangle RQP \; ||| \; \triangle RST$

4. $ABCD$ is a parallelogram with diagonals intersecting at F. FE is drawn parallel to CD. AC is produced to P such that $PC = 2AC$ and AD is produced to Q such that $DQ = 2AD$.

   [NCS, Paper 3, November 2011]

   a) Show that E is the midpoint of AD.
   b) Prove $PQ \parallel FE$.
   c) If PQ is 60 cm, calculate the length of FE.

5. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 299Z 2. 29B2 3. 29B3 4. 29B4

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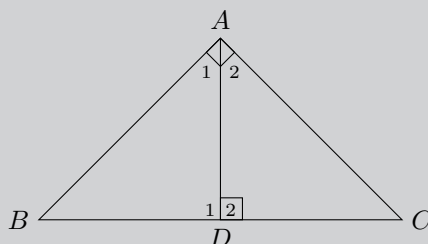
8.6 Pythagorean theorem EMCJH

Many different methods of proving the theorem of Pythagoras have been formulated over the years. Similarity of triangles is one method that provides a neat proof of this important theorem.

Theorem: Pythagorean theorem

STATEMENT

The square on the hypotenuse of a right-angled triangle is equal to the sum of the squares on the other two sides.

(Reason: Pythagoras or right-angled $\triangle$s)

Given:
$\triangle ABC$ with $\hat{A} = 90°$

Required to prove:
$BC^2 = AB^2 + AC^2$

PROOF

Construction: draw $AD \perp BC$.

$$\hat{C} + \hat{A}_2 = 90° \quad (\angle\text{s of } \triangle CAD)$$
$$\hat{A}_1 + \hat{A}_2 = 90° \quad (\text{given})$$
$$\therefore \hat{A}_1 = \hat{C}$$

$$\hat{A}_1 + \hat{B} = 90° \quad (\angle\text{s of } \triangle ABD)$$
$$\therefore \hat{B} = \hat{A}_2$$

$$\hat{D}_1 = \hat{D}_2 = \hat{A} = 90° \quad (\text{construction})$$

$$\therefore \triangle ABD \ ||| \ \triangle CBA \ ||| \ \triangle CAD \quad (\text{AAA})$$

$$\therefore \frac{AB}{BC} = \frac{BD}{AB} \quad (\triangle ABD \ ||| \ \triangle CBA)$$

$$AB^2 = BD \times BC$$

$$\text{Similarly } \frac{AC}{CB} = \frac{DC}{AC} \quad (\triangle ABD \ ||| \ \triangle CBA)$$

$$AC^2 = CB \times DC$$

$$\begin{aligned}\therefore AC^2 + AB^2 &= (BD \times BC) + (CB \times DC) \\ &= BC(BD + DC) \\ &= BC(BC) \\ &= BC^2\end{aligned}$$

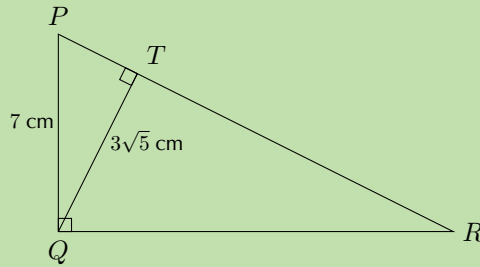
$$\therefore BC^2 = AC^2 + AB^2$$

Converse: theorem of Pythagoras

If the square of one side of a triangle is equal to the sum of the squares of the other two sides of the triangle, then the angle included by these two sides is a right angle.

Worked example 8: Theorem of Pythagoras

QUESTION

In $\triangle PQR$, $P\hat{Q}R = 90°$ and $QT \perp PR$. If $PQ = 7$ cm and $QT = 3\sqrt{5}$ cm, determine PR and QR (correct to the nearest integer).

SOLUTION

Step 1: Use the theorem of Pythagoras to determine PT

$$\begin{aligned} \text{In } \triangle PTQ, \quad PT^2 &= PQ^2 - QT^2 \qquad \text{(Pythagoras)} \\ &= 7^2 - \left(3\sqrt{5}\right)^2 \\ &= 49 - 45 \\ \therefore PT &= \sqrt{4} \\ &= 2 \text{ cm} \end{aligned}$$

Step 2: Use proportionality to determine PR and QR

$$\begin{aligned} P\hat{Q}R &= 90° && \text{(given)} \\ QT &\perp PR && \text{(given)} \\ \therefore \quad \triangle PQT \;|||\; \triangle QRT \;|||\; \triangle PRQ & && \text{(right-angled } \triangle\text{s)} \\ \therefore \frac{QT}{TR} &= \frac{PT}{QT} && (\triangle QRT \;|||\; \triangle PQT) \\ \therefore QT^2 &= TR \,.\, PT \end{aligned}$$

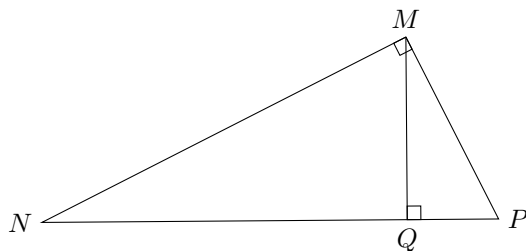
$$\begin{aligned} \left(3\sqrt{5}\right)^2 &= TR \,.\, 2 \\ \frac{45}{2} &= TR \end{aligned}$$

$$\begin{aligned} \text{And } PR &= PT + TR \\ &= \frac{45}{2} + 2 \\ &= 25 \text{ cm} && \text{(to nearest integer)} \end{aligned}$$

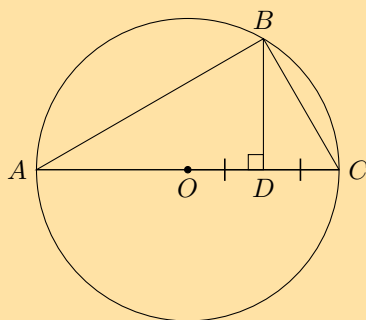
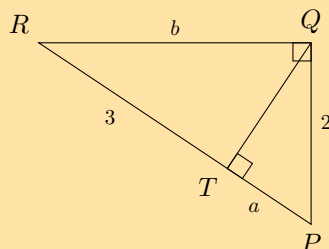
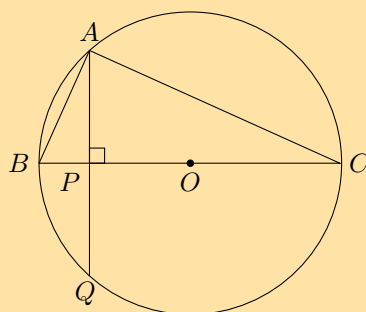
$$\begin{aligned} \text{In } \triangle PQR, \quad QR^2 &= PR^2 - PQ^2 && \text{(Pythagoras)} \\ &= 25^2 - 7^2 \\ \therefore QR &= \sqrt{576} \\ &= 24 \text{ cm} \end{aligned}$$

Step 3: Write the final answer

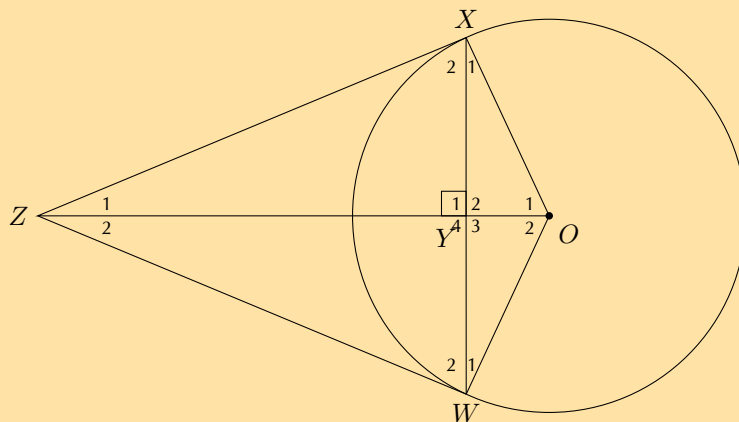
$PR = 25$ cm and $QR = 24$ cm

For any right-angled $\triangle MNP$, if MQ is drawn perpendicular to NP, then:

$$\begin{aligned}
\triangle MNQ \;|||\; \triangle PMQ &\implies MQ^2 = NQ\,.\,PQ \\
\triangle MPQ \;|||\; \triangle NPM &\implies MP^2 = QP\,.\,NP \\
\triangle MNQ \;|||\; \triangle PNM &\implies MN^2 = PN\,.\,QN
\end{aligned}$$

Exercise 8 – 9: Theorem of Pythagoras

1. B is a point on circle with centre O. $BD \perp AC$ and D is the midpoint of radius OC.
   If the diameter of the circle is 24 cm, find BD.
   Leave answer in simplified surd form.

2. In $\triangle PQR$, $RQ \perp QP$ and $QT \perp RP$. $PQ = 2$ units, $QR = b$ units, $RT = 3$ units and $TP = a$ units. Determine a and b, giving reasons.

3. Chord AQ of circle with centre O cuts BC at right angles at point P.

a) Why is $\triangle ABP \ ||| \ \triangle CBA$?

b) If $AB = \sqrt{6}$ units and $PO = 2$ units, calculate the radius of the circle.

4. In the diagram below, XZ and WZ are tangents to the circle with centre O and $X\hat{Y}Z = 90°$.

a) Show that $XY^2 = OY \ . \ YZ$.

b) Prove that $\frac{OY}{YZ} = \frac{OW^2}{WZ^2}$.

5. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 29B5 2. 29B6 3. 29B7 4. 29B8

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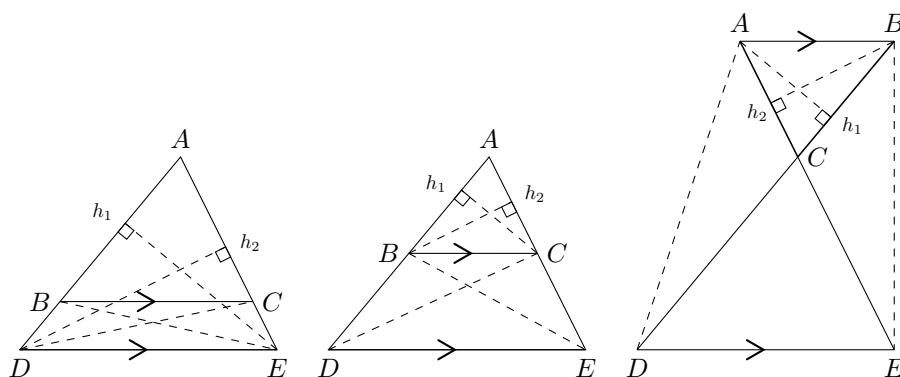
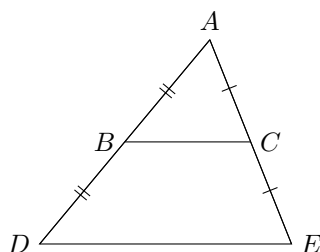
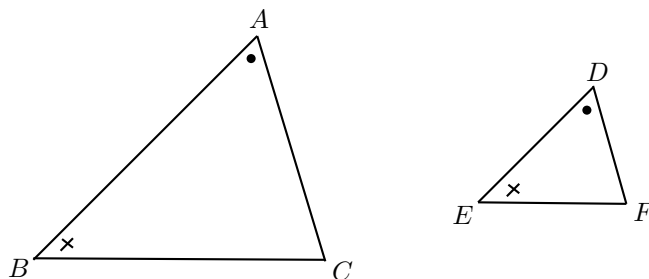
8.7 Summary

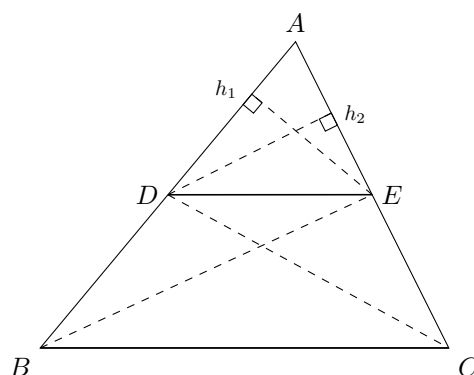
- A ratio describes the relationship between two quantities which have the same units.

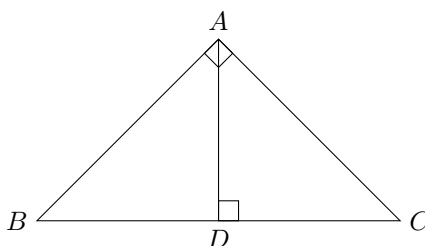
$$x : y \quad \text{or} \quad \frac{x}{y} \quad \text{or} \quad x \ \text{to} \ y$$

- If two or more ratios are equal to each other $\left(\frac{m}{n} = \frac{p}{q}\right)$, then m and n are in the same proportion as p and q.
- A polygon is a plane, closed shape consisting of three or more line segments.
- Triangles with equal heights have areas which are proportional to their bases.
- Triangles with equal bases and between the same parallel lines are equal in area.
- Triangles on the same side of the same base and equal in area lie between parallel lines.
- A line drawn parallel to one side of a triangle divides the other two sides of the triangle in the same proportion.

$$\frac{AD}{DB} = \frac{AE}{EC} \quad \text{and} \quad \frac{AB}{AD} = \frac{AC}{AE} \quad \text{and} \quad \frac{CD}{BD} = \frac{CE}{AE}$$

- **Converse: proportion theorem**
  If a line divides two sides of a triangle in the same proportion, then the line is parallel to the third side.
- **Special case: the mid-point theorem**
  The line joining the mid-points of two sides of a triangle is parallel to the third side and equal to half the length of the third side.
  If $AB = BD$ and $AC = CE$, then $BC \parallel DE$ and $BC = \frac{1}{2}DE$.

Converse: the mid-point theorem
The line drawn from the mid-point of one side of a triangle parallel to another side, bisects the third side of the triangle.
If $AB = BD$ and $BC \parallel DE$, then $AC = CE$.

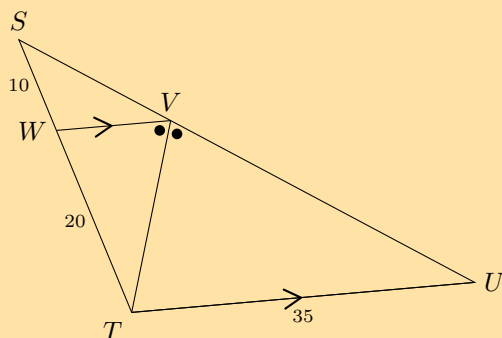
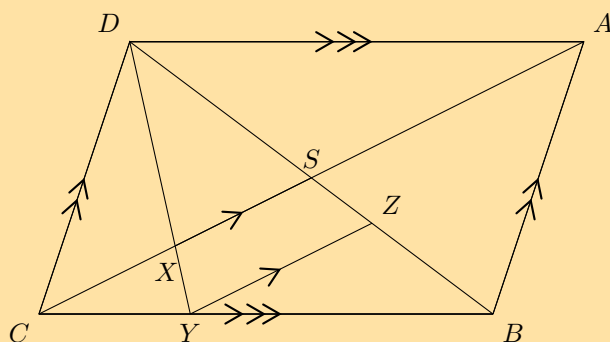
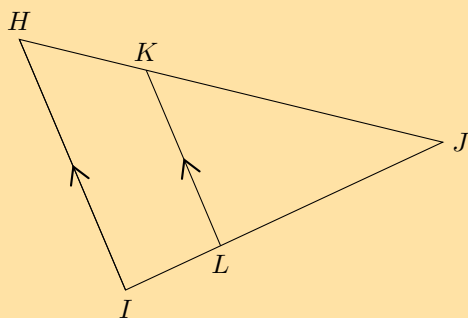
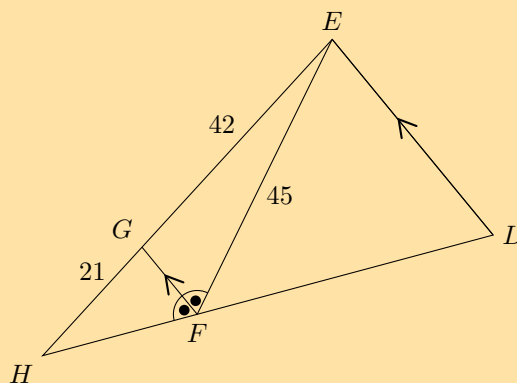
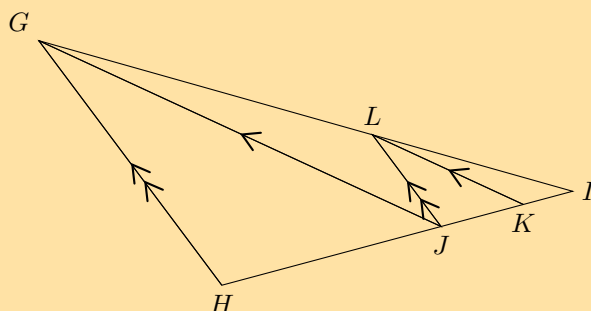
- Polygons are similar if they are the same shape but differ in size. One polygon is an enlargement of the other.
- Two polygons with the same number of sides are similar when:
  1. All pairs of corresponding angles are equal, **and**
  2. All pairs of corresponding sides are in the same proportion.
- If two triangles are equiangular, then the triangles similar.

$$\triangle ABC \ ||| \ \triangle DEF$$

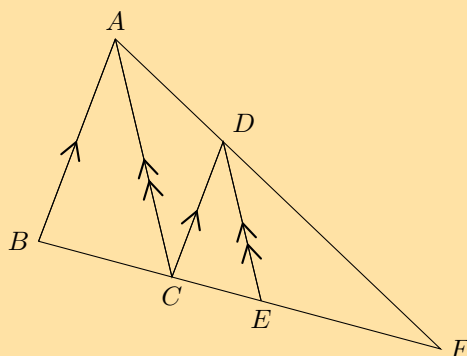
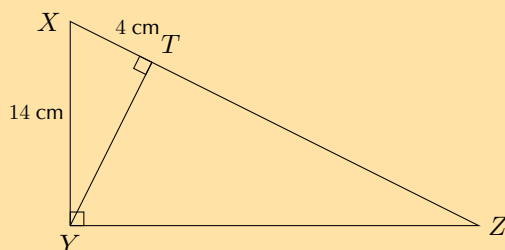
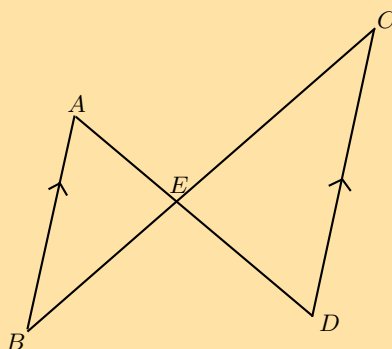
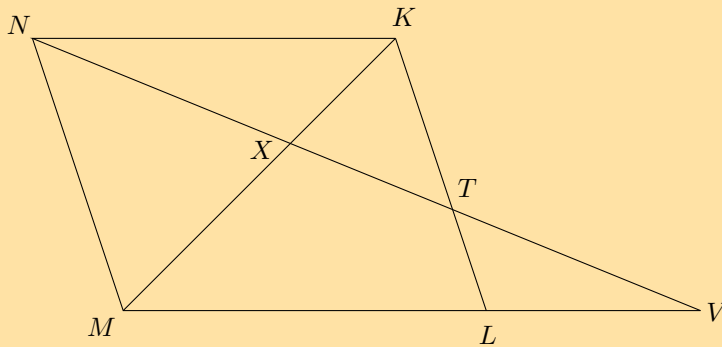
- Triangles with sides in proportion are equiangular and therefore similar.

$$\triangle ADE \ ||| \ \triangle ABC$$

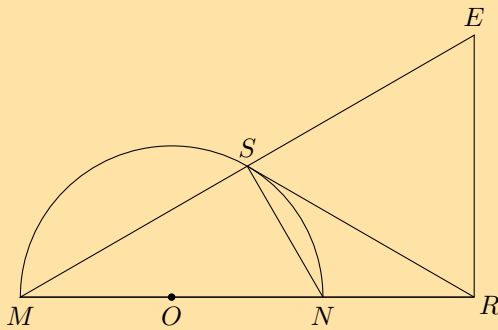
- The square on the hypotenuse of a right-angled triangle is equal to the sum of the squares on the other two sides.

$$BC^2 = AC^2 + AB^2$$

- **Converse: theorem of Pythagoras**
  If the square of one side of a triangle is equal to the sum of the squares of the other two sides of the triangle, then the angle included by these two sides is a right angle.

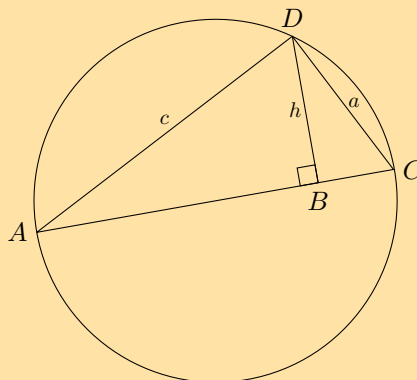
Exercise 8 – 10: End of chapter exercises

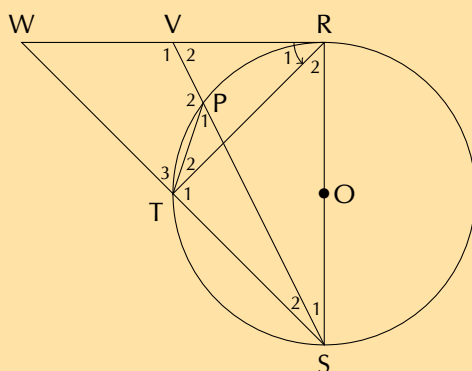
1. Calculate SV

2. $\frac{CB}{YB} = \frac{3}{2}$. Find $\frac{DS}{SZ}$.

3. Using the following figure and lengths, find IJ and KJ (correct to one decimal place).
   $HI = 20$ m, $KL = 14$ m, $JL = 18$ m and $HJ = 32$ m.

4. Find FH in the following figure.

5. In $\triangle GHI$, $GH \parallel LJ$, $GJ \parallel LK$ and $\frac{JK}{KI} = \frac{5}{3}$. Determine $\frac{HJ}{KI}$.

6. $BF = 25$ m, $AB = 13$ m, $AD = 9$ m, $DF = 18$ m.

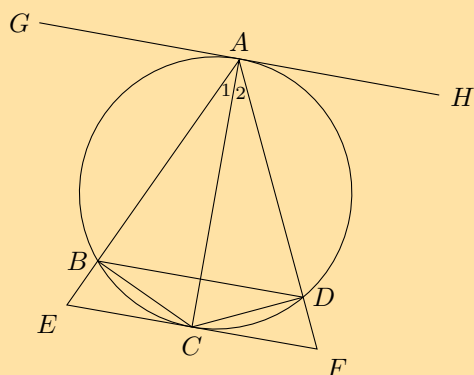
   Calculate the lengths of BC, CF, CD, CE and EF, and find the ratio $\frac{DE}{AC}$.

7. In $\triangle XYZ$, $X\hat{Y}Z = 90°$ and $YT \perp XZ$. If $XY = 14$ cm and $XT = 4$ cm, determine XZ and YZ (correct to two decimal places).

8. Given the following figure with the following lengths, find AE, EC and BE.
   $BC = 15$ cm, $AB = 4$ cm, $CD = 18$ cm, and $ED = 9$ cm.

9. $NKLM$ is a parallelogram with T on KL.
NT produced meets ML produced at V. NT intercepts MK at X.

a) Prove that $\frac{XT}{NX} = \frac{XK}{MX}$.
b) Prove $\triangle VXM \ ||| \ \triangle NXK$.
c) If $XT = 3$ cm and $TV = 4$ cm, calculate NX.

10. MN is a diameter of circle O. MN is produced to R so that $MN = 2NR$.
RS is a tangent to the circle and $ER \perp MR$. MS produced meets RE at E.

Prove that:

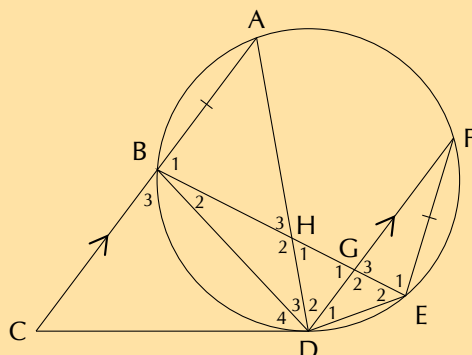
a) $SNRE$ is a cyclic quadrilateral
b) $RS = RE$
c) $\triangle MSN \ ||| \ \triangle MRE$
d) $\triangle RSN \ ||| \ \triangle RMS$
e) $RE^2 = RN.RM$

11. AC is a diameter of circle ADC. $DB \perp AC$.
$AC = d, AD = c, DC = a$ and $DB = h$.

a) Prove that $h = \frac{ac}{d}$.
b) Hence, deduce that $\frac{1}{h^2} = \frac{1}{a^2} + \frac{1}{c^2}$.

12. RS is a diameter of the circle with centre O. Chord ST is produced to W. Chord SP produced meets the tangent RW at V. $\hat{R}_1 = 50°$. [NCS, Paper 3, November 2011]

a) Calculate the size of $W\hat{R}S$.
b) Find $\hat{W}$.
c) Determine the size of $\hat{P}_1$.
d) Prove that $\hat{V}_1 = P\hat{T}S$.

13. $ABCD$ is a cyclic quadrilateral and $BC = CD$.
ECF is a tangent to the circle at C. ABE and ADF are straight lines.

Prove:

a) AC bisects $E\hat{A}F$
b) $BD \parallel EF$
c) $\triangle ADC \; ||| \; \triangle CBE$
d) $DC^2 = AD.BE$

14. CD is a tangent to circle $ABDEF$ at D. Chord AB is produced to C. Chord BE cuts chord AD in H and chord FD in G. $AC \parallel FD$ and $E = AB$. Let $D_4 = x$ and $D_1 = y$. [NCS, Paper 3, November 2011]

a) Determine THREE other angles that are each equal to x.
b) Prove that $\triangle$BHD $|||$ $\triangle$FED.
c) Hence, or otherwise, prove that $AB \,.\, BD = FD \,.\, BH$.

15. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 29B9	2. 29BB	3. 29BC	4. 29BD	5. 29BF	6. 29BG
7. 29BH	8. 29BJ	9. 29BK	10. 29BM	11. 29BN	12. 29BP
13. 29BQ	14. 29BR				

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CHAPTER

Statistics

9.1 Revision EMCJK

Terminology EMCJM

Measures of central tendency:

Provide information on the data values at the centre of the data set.

- The **mean** is the 'average' value of a data set. It is calculated as

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

  where the x_i are the data and n is the number of data entries. We read $\bar{x}$ as "x bar".

- The **median** is the middle value of an ordered data set. To find the median, we first sort the data in ascending or descending order and then pick out the value in the middle of the sorted list. If the middle is in between two values, the median is the average of those two values.

Measures of dispersion:

Tell us how spread out a data set is. If a measure of dispersion is small, the data are clustered in a small region. If a measure of dispersion is large, the data are spread out over a large region.

- The **range** is the difference between the maximum and minimum values in the data set.

- The **inter-quartile range** is the difference between the first and third quartiles of the data set. The quartiles are computed in a similar way to the median. The median is halfway into the ordered data set and is sometimes also called the second quartile. The first quartile is one quarter of the way into the ordered data set, whereas the third quartile is three quarters of the way into the ordered data set.

  If you begin numbering your ordered data set with the number 1, the formulae for the location of each quartile are as follows:

$$\text{Location of } Q_1 = \frac{1}{4}(n-1)+1$$
$$\text{Location of } Q_2 = \frac{1}{2}(n-1)+1$$
$$\text{Location of } Q_3 = \frac{3}{4}(n-1)+1$$

- The **variance** of the data is the average squared distance between the mean and each data value.
  The variance of the data is

$$\sigma^2 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n}$$

  in a population of n elements, $\{x_1; x_2; \ldots; x_n\}$, with a mean of $\bar{x}$.

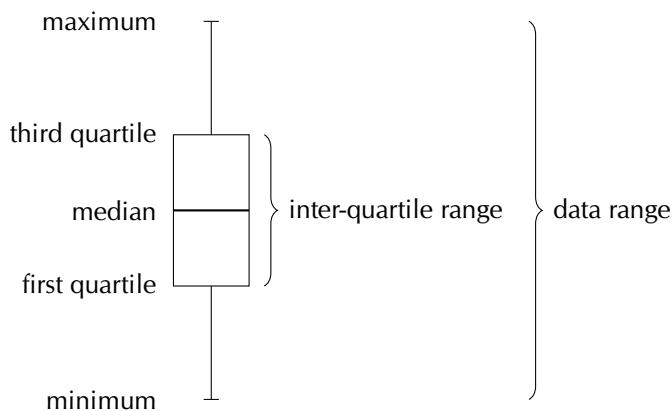
- The **standard deviation** measures how spread out the values in a data set are around the mean. More precisely, it is a measure of the average distance between the values of the data in the set and the mean.
  The standard deviation of the data is

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n}}$$

  in a population of n elements, $\{x_1; x_2; \ldots; x_n\}$, with a mean of $\bar{x}$.

The **five number summary** combines a measure of central tendency, the median, with measures of dispersion, namely the range and the inter-quartile range. More precisely, the five number summary is written in the following order:

- minimum;
- first quartile;
- median;
- third quartile;
- maximum.

The five number summary is often presented visually using a **box and whisker diagram**, illustrated below.

See video: 29BS at www.everythingmaths.co.za

Worked example 1: Five number summary

QUESTION

Draw a box and whisker diagram for the following data set:

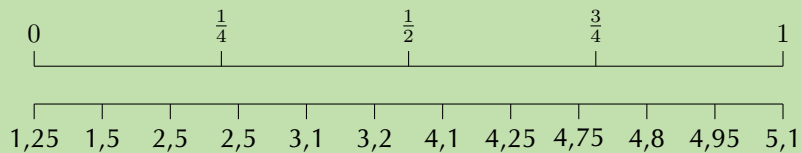
$$1{,}25\ ;\ 1{,}5\ ;\ 2{,}5\ ;\ 2{,}5\ ;\ 3{,}1\ ;\ 3{,}2\ ;\ 4{,}1\ ;\ 4{,}25\ ;\ 4{,}75\ ;\ 4{,}8\ ;\ 4{,}95\ ;\ 5{,}1$$

SOLUTION

Step 1: Determine the minimum and maximum

Since the data set is already ordered, we can read off the minimum as the first value ($1{,}25$) and the maximum as the last value ($5{,}1$).

Step 2: Determine the quartiles

There are 12 values in the data set. We can use the figure below or the formulae to determine where the quartiles are located.

Using the figure above we can see that the median is between the sixth and seventh values. We can confirm this using the formula:

$$\begin{aligned}\text{Location of } Q_2 &= \frac{1}{2}(n-1)+1 \\ &= \frac{1}{2}(11)+1 \\ &= 6{,}5\end{aligned}$$

Therefore, the value of the median is

$$\frac{3{,}2+4{,}1}{2} = 3{,}65$$

The first quartile lies between the third and fourth values. We can confirm this using the formula:

$$\begin{aligned}\text{Location of } Q_1 &= \frac{1}{4}(n-1)+1 \\ &= \frac{1}{4}(11)+1 \\ &= 3{,}75\end{aligned}$$

Therefore, the value of the first quartile is

$$Q_1 = \frac{2{,}5+2{,}5}{2} = 2{,}5$$

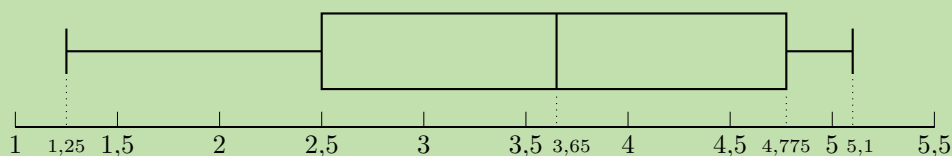
The third quartile lies between the ninth and tenth values. We can confirm this using the formula:

$$\begin{aligned} \text{Location of } Q_3 &= \frac{3}{4}(n-1)+1 \\ &= \frac{3}{4}(11)+1 \\ &= 9{,}25 \end{aligned}$$

Therefore, the value of the third quartile is

$$Q_3 = \frac{4{,}75 + 4{,}8}{2} = 4{,}775$$

Step 3: Draw the box and whisker diagram

We now have the five number summary as $(1{,}25;\ 2{,}5;\ 3{,}65;\ 4{,}775;\ 5{,}1)$. The box and whisker diagram representing the five number summary is given below.

See video: 29BT at www.everythingmaths.co.za

Worked example 2: Variance and standard deviation

QUESTION

You flip a coin 100 times and it lands on heads 44 times. You then use the same coin and do another 100 flips. This time in lands on heads 49 times. You repeat this experiment a total of 10 times and get the following results for the number of heads.

$$\{44; 49; 52; 62; 53; 48; 54; 49; 46; 51\}$$

For the data set above:

- Calculate the mean.
- Calculate the variance and standard deviation using a table.
- Confirm your answer for the variance and standard deviation using a calculator.

SOLUTION

Step 1: Calculate the mean

The formula for the mean is

$$\overline{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

In this case, we sum the data and divide by 10 to get $\overline{x} = 50{,}8$.

Step 2: Calculate the variance using a table

The formula for the variance is

$$\sigma^2 = \frac{\sum_{i=1}^{n} (x_i - \overline{x})^2}{n}$$

We first subtract the mean from each data point and then square the result.

x_i	44	49	52	62	53	48	54	49	46	51
$x_i - \overline{x}$	$-6{,}8$	$-1{,}8$	$1{,}2$	$11{,}2$	$2{,}2$	$-2{,}8$	$3{,}2$	$-1{,}8$	$-4{,}8$	$0{,}2$
$(x_i - \overline{x})^2$	$46{,}24$	$3{,}24$	$1{,}44$	$125{,}44$	$4{,}84$	$7{,}84$	$10{,}24$	$3{,}24$	$23{,}04$	$0{,}04$

The variance is the sum of the last row in this table divided by 10, so $\sigma^2 = 22{,}56$.

Step 3: Calculate the variance using a calculator

Using the SHARP EL-531VH calculator:

Using your calculator, change the mode from normal to "Stat x ". Do this by pressing [2ndF] and then 1. This mode enables you to type in univariate data.

Key in the data, row by row:

Enter:	Press:	See:
44	**DATA**	n = 1
49	**DATA**	n = 2
52	**DATA**	n = 3
62	**DATA**	n = 4
53	**DATA**	n = 5
48	**DATA**	n = 6
54	**DATA**	n = 7
49	**DATA**	n = 8
46	**DATA**	n = 9
51	**DATA**	n = 10

Note: The [**DATA**] button is the same as the [M+] button.

Get the value for σ_x:

Press:	Press:	See:
RCL	σx	$\sigma x = \pm 4{,}75$

$\therefore \sigma_x = \pm 4{,}75$ and $\sigma_x{}^2 = (4{,}75)^2 = 22{,}56$

Using the CASIO fx-82ZA PLUS calculator:

Switch on the calculator. Press [MODE] and then select STAT by pressing [2]. The following screen will appear:

1:	$1 - VAR$	2:	$A + BX$
3:	$_{+}CX^2$	4:	lnX
5:	eX	6:	$A.BX$
7:	$A.XB$	8:	$1/X$

Now press [1] for variance and standard deviation. Your screen should look something like this:

	X
1	
2	
3	

Press [44] and then [=] to enter the first x-value under x. Then enter the other values in the same way.

	X
1	44
2	49
3	52

Then press [AC]. The screen clears but the data remains stored.

Now press [SHIFT][1] to get the stats computations screen shown below.

1:	Type	2:	Data
3:	Sum	4:	Var
5:	MinMax		

Choose variance by pressing [4].

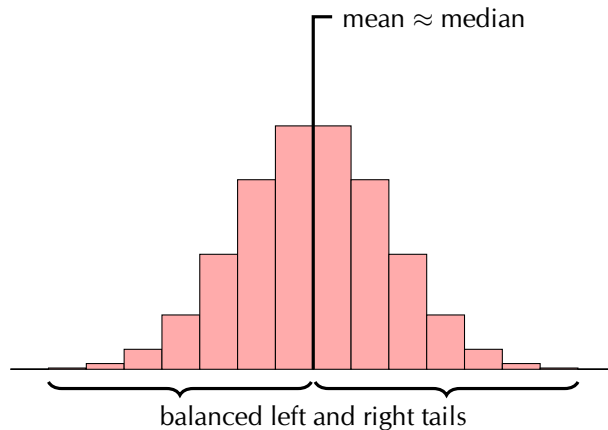
1:	n	2:	$\bar{x}$
3:	σx	4:	sx

Get the value for σ_x:

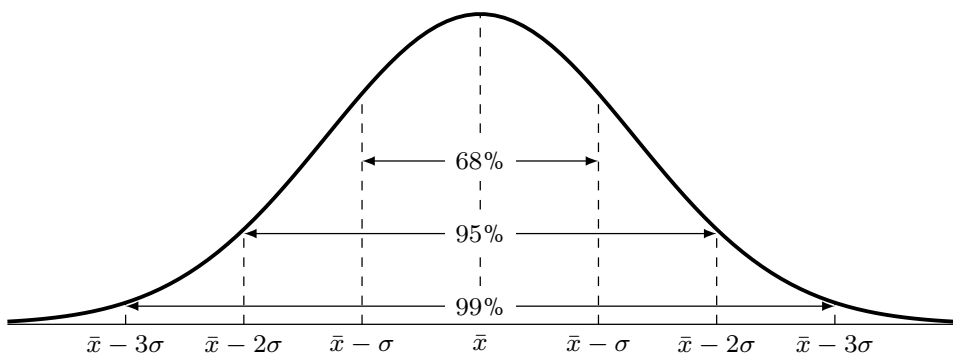
Press [3] and [=] to get the value of $\sigma x \therefore \sigma_x = \pm 4{,}75$ and ${\sigma_x}^2 = (4{,}75)^2 = 22{,}56$

Symmetric and skewed data

Last year you learnt about three shapes of data distribution: symmetric, left skewed and right skewed.

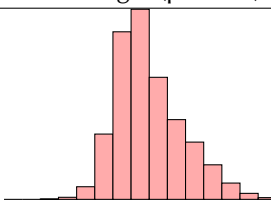
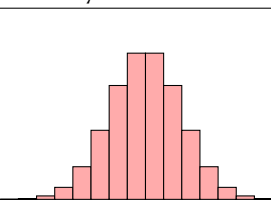
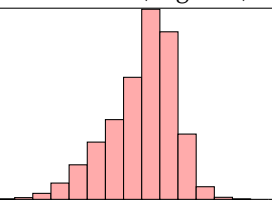
A *symmetric* distribution is one where the left and right hand sides of the distribution are roughly equally balanced around the mean. The histogram below shows a typical symmetric distribution.

For symmetric distributions, the mean is approximately equal to the median and the left and right tails are equally balanced, meaning that they have about the same length.

If large numbers of data are collected from a population, the graph will often have a bell shape. If the data was, say, examination results, a few learners usually get very high marks, a few very low marks and most get a mark in the middle range. This is a common type of symmetric data known as a *normal* distribution. We say a distribution is normal if

- the mean, median and mode are equal.
- it is symmetric around the mean.
- 68% of the sample lies within one standard deviation of the mean, 95% within two standard deviations and 99% within three standard deviations of the mean.

See video: 29BV at www.everythingmaths.co.za

What happens if the test was very easy or very difficult? Then the distribution may not be symmetrical. If extremely high or extremely low scores are added to a distribution, then the mean and median tend to shift towards these scores and the curve becomes skewed.

If the test was very difficult, the mean and median scores are shifted to the left. In this case, we say the distribution is *positively skewed*, or *skewed right*.

A distribution that is skewed right has the following characteristics:

- the mean is typically more than the median;
- the tail of the distribution is longer on the right hand side than on the left hand side; and
- the median is closer to the first quartile than to the third quartile.

If the test was very easy, then many learners would get high scores, and the mean and median of the distribution would be shifted to the right. We say the distribution is *negatively skewed*, or *skewed left*.

A distribution that is skewed right has the following characteristics:

- the mean is typically less than the median;
- the tail of the distribution is longer on the left hand side than on the right hand side; and
- the median is closer to the third quartile than to the first quartile.

The table below summarises the different categories visually.

Worked example 3: Skewed and symmetric data

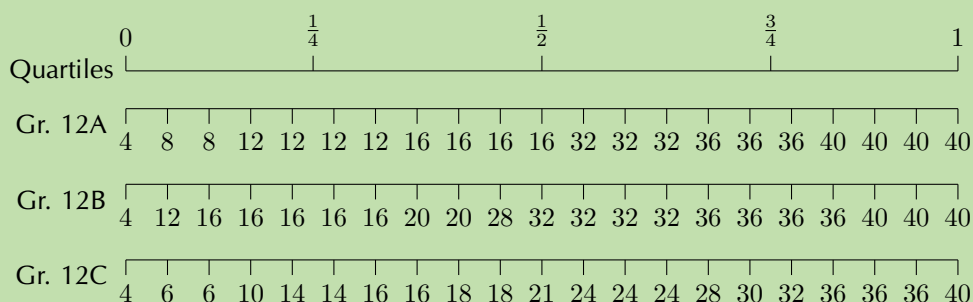
QUESTION

Three Matric classes wrote a Mathematics test. The test is out of 40 marks and each class has 21 learners. The results of the test are shown in the table on the next page.

Gr. 12A	Gr. 12B	Gr. 12C
4	4	4
8	12	6
8	16	6
12	16	10
12	16	14
12	16	14
12	16	16
16	20	16
16	20	18
16	28	18
16	32	21
32	32	24
32	32	24
32	32	24
36	36	28
36	36	30
36	36	32
40	36	36
40	40	36
40	40	36
40	40	40

1. For each class, determine the five number summary and draw a box and whisker diagram on the same set of axes using an appropriate scale.
2. Determine the mean and standard deviation for each class.
3. Comparing the mean and median values for each class, comment on the distribution of the test marks for each class.

SOLUTION

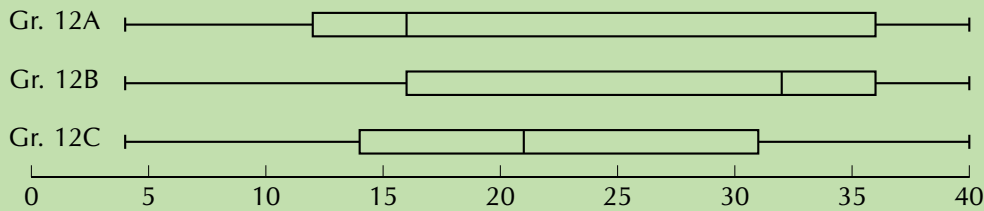
1. First, we order the data from smallest to largest. This has already been done for us. Then, we divide our data into quartiles:

The minimum of each data set is 4. The maximum of each data set is 40.

Since there are 21 values in the data set, the median lies on the eleventh mark, making it equal to 16 for Gr. 12A, 32 for Gr. 12B and 21 for Gr. 12C.

The first quartile lies between the fifth and sixth values, making it equal to 12 for Gr. 12A, 16 for Gr. 12B and 14 for Gr. 12C.

The third quartile lies between the 16^{th} and 17^{th} values, making it equal to 36 for Gr. 12A and Gr. 12B, and $\frac{30+32}{2} = 31$ for Gr. 12C.

Therefore, we are able to formulate the following five number summaries and subsequent box and whisker plots:

- Gr. 12A $= [4; 12; 16; 36; 40]$
- Gr. 12B $= [4; 16; 32; 36; 40]$
- Gr. 12C $= [4; 14; 21; 31; 40]$

2. Gr. 12A:

$$\text{mean } (\bar{x}) = \frac{496}{21} = 23{,}6$$

$$\text{standard deviation } (\sigma) = \sqrt{\frac{\sum (x - \bar{x})^2}{n}} = \pm 12{,}70$$

Gr. 12B:

$$\text{mean } (\bar{x}) = \frac{556}{21} = 26{,}5$$

$$\text{standard deviation } (\sigma) = \sqrt{\frac{\sum (x - \bar{x})^2}{n}} = \pm 10{,}65$$

Gr. 12C:

$$\text{mean } (\bar{x}) = \frac{453}{21} = 21{,}6$$

$$\text{standard deviation } (\sigma) = \sqrt{\frac{\sum (x - \bar{x})^2}{n}} = \pm 10{,}54$$

3. If the mean is greater than the median, the data is typically positively skewed and if the mean is less than the median, the data is typically negatively skewed.

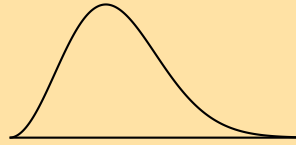
Gr. 12A: mean $-$ median $= 23{,}6 - 16 = 7{,}6$. The marks for 12A are therefore positively skewed, meaning that there were many low marks in the class with the high marks being more spread out.

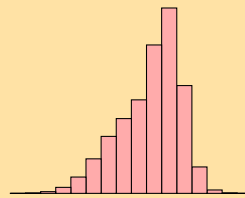
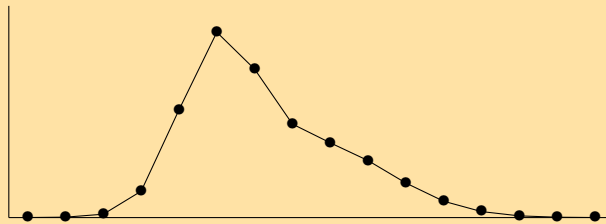
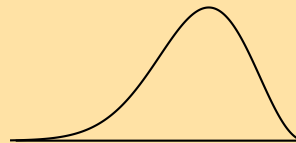
Gr. 12B: mean $-$ median $= 26{,}5 - 32 = -5{,}5$. The marks for 12B are therefore negatively skewed, meaning that there were many high marks in the class with the low marks being more spread out.

Gr. 12C: mean $-$ median $= 21{,}6 - 21 = 0{,}6$. The marks for 12C are therefore normally distributed, meaning that there are as many low marks in the class as there are high marks.

Exercise 9 – 1: Revision

1. State whether each of the following data sets are symmetric, skewed right or skewed left.

   a) A data set with this distribution:

   b) A data set with this box and whisker plot:

   c) A data set with this histogram:

   d) A data set with this frequency polygon:

   e) A data set with this distribution:

   f) The following data set:
   $105 ; 44 ; 94 ; 149 ; 83 ; 178 ; -4 ; 112 ; 50 ; 188$

2. For the following data sets:

   - Determine the mean and five number summary.
   - Draw the box and whisker plot.
   - Determine the skewness of the data.

   a) $40 ; 45 ; 12 ; 6 ; 9 ; 16 ; 11 ; 7 ; 35 ; 7 ; 31 ; 3$

   b) $65 ; 100 ; 99 ; 21 ; 8 ; 27 ; 21 ; 31 ; 33 ; 31 ; 38 ; 16$

   c) $65 ; 57 ; 77 ; 92 ; 77 ; 58 ; 90 ; 46 ; 11 ; 81$

   d) $1 ; 99 ; 76 ; 76 ; 50 ; 74 ; 83 ; 91 ; 41 ; 17 ; 33$

   e) $0{,}5; -0{,}9 ; -1{,}8 ; 3 ; -0{,}2 ; -5{,}2 ; -1{,}8 ; 0{,}1 ; -1{,}7 ; -2 ; 2{,}2 ; 0{,}5; -0{,}5$

   f) $86 ; 64 ; 25 ; 71 ; 54 ; 44 ; 97 ; 31 ; 78 ; 46 ; 60 ; 86$

3. For the following data sets:
   - Determine the mean.
   - Use a table to determine the variance and the standard deviation.
   - Determine the percentage of data points within one standard deviation of the mean. Round your answer to the nearest percentage point.

   a) $\{9{,}1; 0{,}2; 2{,}8; 2{,}0; 10{,}0; 5{,}8; 9{,}3; 8{,}0\}$
   b) {9; 5; 1; 3; 3; 5; 7; 4; 10; 8}
   c) {81; 22; 63; 12; 100; 28; 54; 26; 50; 44; 4; 32}

4. Use a calculator to determine the
   - mean,
   - variance,
   - and standard deviation

   of the following data sets:

   a) $8 ; 3 ; 10 ; 7 ; 7 ; 1 ; 3 ; 1 ; 3 ; 7$
   b) $4 ; 4 ; 13 ; 9 ; 7 ; 7 ; 2 ; 5 ; 15 ; 4 ; 22 ; 11$
   c) $4{,}38 ; 3{,}83 ; 4{,}99 ; 4{,}05 ; 2{,}88 ; 4{,}83 ; 0{,}88 ; 5{,}33 ; 3{,}49 ; 4{,}10$
   d) $4{,}76 ; -4{,}96 ; -6{,}35 ; -3{,}57 ; 0{,}59 ; -2{,}18 ; -4{,}96 ; -3{,}57 ; -2{,}18 ; 1{,}98$
   e) $7 ; 53 ; 29 ; 42 ; 12 ; 111 ; 122 ; 79 ; 83 ; 5 ; 69 ; 45 ; 23 ; 77$

5. Xolani surveyed the price of a loaf of white bread at two different supermarkets. The data, in rands, are given below.

Supermarket A	3,96	3,76	4,00	3,91	3,69	3,72
Supermarket B	3,97	3,81	3,52	4,08	3,88	3,68

   a) Find the mean price at each supermarket and then state which supermarket has the lower mean.
   b) Find the standard deviation of each supermarket's prices.
   c) Which supermarket has the more consistently priced white bread? Give reasons for your answer.

6. The times for the 8 athletes who swam the 100 m freestyle final at the 2012 London Olympic Games are shown below. All times are in seconds.
   $47{,}52 ; 47{,}53 ; 47{,}80 ; 47{,}84 ; 47{,}88 ; 47{,}92 ; 48{,}04 ; 48{,}44$

   a) Calculate the mean time.
   b) Calculate the standard deviation for the data.
   c) How many of the athletes' times are more than one standard deviation away from the mean?

7. The following data set has a mean of $14{,}7$ and a variance of $10{,}01$.

$$18 ; \ 11 ; \ 12 ; \ a ; \ 16 ; \ 11 ; \ 19 ; \ 14 ; \ b ; \ 13$$

   Calculate the values of a and b.

8. The height of each learner in a class was measured and it was found that the mean height of the class was $1{,}6$ m. At the time, three learners were absent. However, when the heights of the learners who were absent were included in the data for the class, the mean height did not change.

   If the heights of two of the learners who were absent are $1{,}45$ m and $1{,}63$ m, calculate the height of the third learner who was absent. [NSC Paper 3 Feb-March 2013]

9. There are 184 students taking Mathematics in a first-year university class. The marks, out of 100, in the half-yearly examination are normally distributed with a mean of 72 and a standard deviation of 9. [NSC Paper 3 Feb-March 2013]

   a) What percentage of students scored between 72 and 90 marks?
   b) Approximately how many students scored between 45 and 63 marks?

10. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 29BW 1b. 29BX 1c. 29BY 1d. 29BZ 1e. 29C2 1f. 29C3
2a. 29C4 2b. 29C5 2c. 29C6 2d. 29C7 2e. 29C8 2f. 29C9
3a. 29CB 3b. 29CC 3c. 29CD 4a. 29CF 4b. 29CG 4c. 29CH
4d. 29CJ 4e. 29CK 5. 29CM 6. 29CN 7. 29CP 8. 29CQ
9. 29CR

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9.2 Curve fitting EMCJP

Intuitive curve fitting EMCJQ

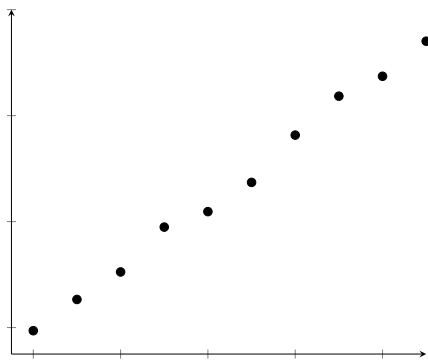
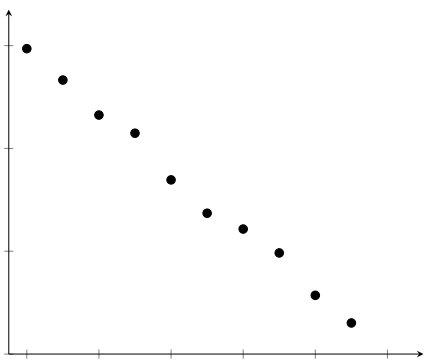
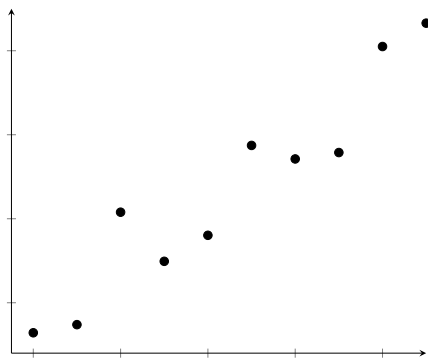
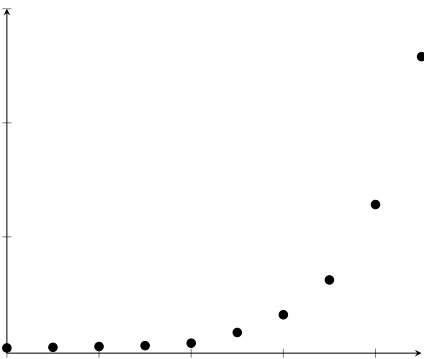
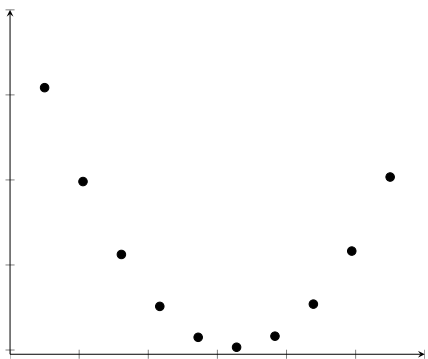
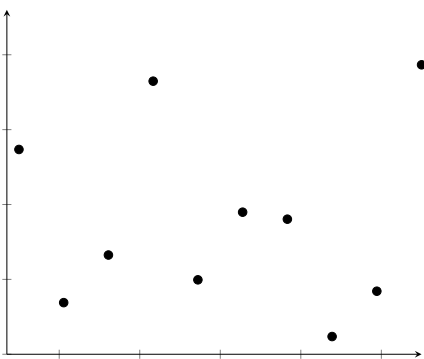
In Grade 11, we used various means, such as histograms, frequency polygons and ogives, to visualise our data. These are very useful tools to depict **univariate** data, i.e. data with only one variable such as the height of learners in a class.

Last year we also learnt about a visual tool called scatter plots. **Scatter plots** are a common way to visualise **bivariate** data, i.e. data with two variables. This allows us to identify the *direction* and *strength* of a relationship between two variables.

We identify the nature of a relationship between two variables by examining if the points on the scatter plot conform to a linear, exponential, quadratic or some other function. The process of fitting functions to data is known as **curve fitting**.

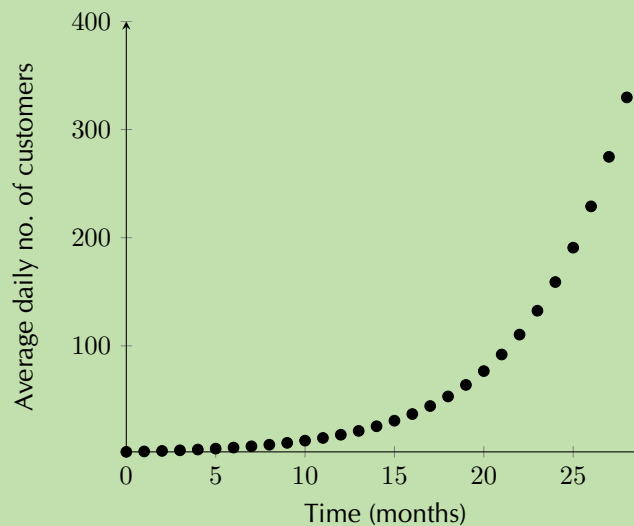
The strength of a relationship can be described as *strong* if the data points conform closely to a function or *weak* if they are further away.

In the case of linear functions, the direction of a relationship is *positive* if high values of one variable occur with high values of the other or *negative* if high values of one variable occur with low values of the other.

The different relationships are summarised below:

Strong, positive linear relationship

Strong, negative linear relationship

Weak, positive linear relationship

Exponential relationship

Quadratic relationship

No relationship

Worked example 4: Intuitive curve fitting

QUESTION

Examine the scatter plot below of data collected from a new shop:

- What are the two variables being compared?
- What type of function best fits the data?
- Is the relationship between the two variables strong or weak?
- Is the relationship between the two variables positive or negative?
- Using your answers above, describe the relationship between the two variables in one sentence.

SOLUTION

- The variables being compared are average daily number of customers and time in months.
- The data fit an exponential function.
- The data points appear to fit the curve close to perfectly, so the relationship can be described as very strong.
- As time increases, the number of customers increases, so the relationship can be described as positive.
- There is a very strong, positive, exponential relationship between average daily customers and time in the new shop.

In the worked example above, by plotting the average daily customers and time data of a new shop on a scatter plot, we were able to identify the relationship between the two variables. Once we know the relationship between two variables, we are able to do another very useful thing we can predict values where no data exist.

DEFINITION: *Interpolation and extrapolation*

When we predict values that fall within the range of our data, this is known as **interpolation**. When we predict the values of a variable beyond the range of our data, this is known as **extrapolation**.

Extrapolation must be done with caution unless it is known that the observed relationship continues beyond the range of our data. For example, an exponential function may look linear if we only have the first few data points available but if we extrapolate far enough beyond the initial data points, our predictions will be inaccurate.

In order to interpolate or extrapolate values, we need to find the equation of the function which best fits the data. For linear data, we draw a straight line through the data which best approximates the available data points. This line is known as the **line of best fit** or trend line. Let us try our hand at this in the following example.

Worked example 5: Fitting by hand

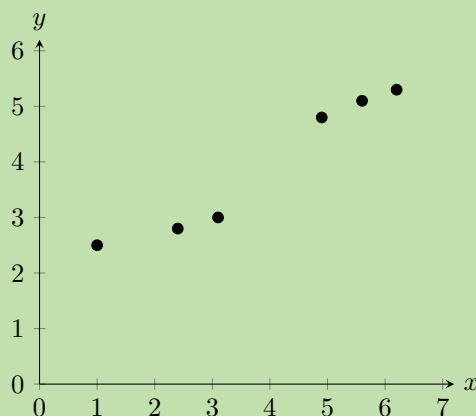
QUESTION

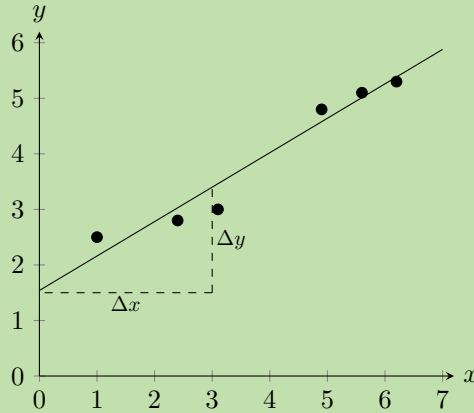
- Use the data below to draw a scatter plot and line of best fit.
- Write down the equation of the line that best seems to fit the data.
- Use your equation to calculate the estimated value for y if $x = 4$.
- Use your equation to calculate the estimated value for x if $y = 6$.

x	1,0	2,4	3,1	4,9	5,6	6,2
y	2,5	2,8	3,0	4,8	5,1	5,3

SOLUTION

Step 1: Draw the graph

1. Choose a suitable scale for the axes.
2. Draw the axes.
3. Plot the points.

Step 2: Drawing the line of best fit

The next step is to draw a straight line which goes as close to as many points as possible. It is generally best to have as many points above the line as below the line.

Step 3: Calculating the equation of the line

The equation of the line is $y = mx + c$

From the graph we have drawn, we estimate the y-intercept to be $1{,}5$. We estimate that $y = 3{,}5$ when $x = 3$. So we have that points $(3; 3{,}5)$ and $(0; 1{,}5)$ lie on the line. The gradient of the line, m, is given by

$$\begin{aligned} m &= \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{3{,}5 - 1{,}5}{3 - 0} \\ &= \frac{2}{3} \end{aligned}$$

So we finally have that the equation of the line of best fit is $y = \frac{2}{3}x + 1{,}5$

Step 4: Calculate the unknown values

The equation of the line is $y = \frac{2}{3}x + 1{,}5$ so in order to find the unknown values, we insert the known values into our equation.

For $x = 4$:

$$\begin{aligned} y &= \frac{2}{3} \cdot 4 + 1{,}5 \\ &= 4{,}17 \end{aligned}$$

Since this x-value is within the data range, this is **interpolation**.

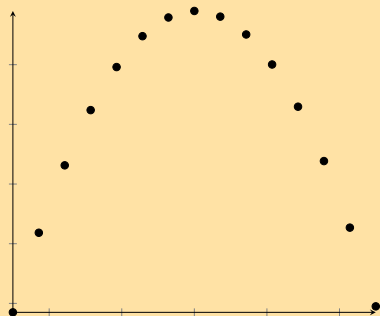
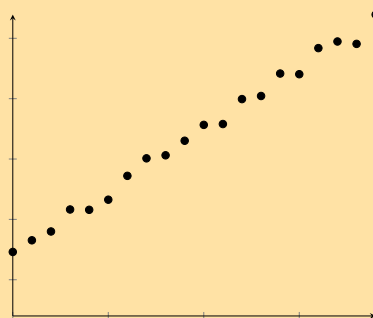
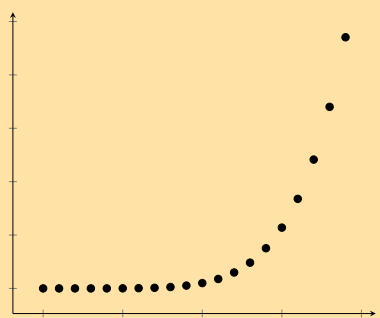
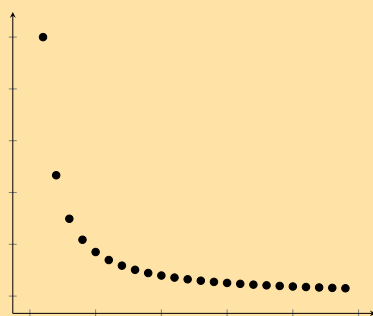
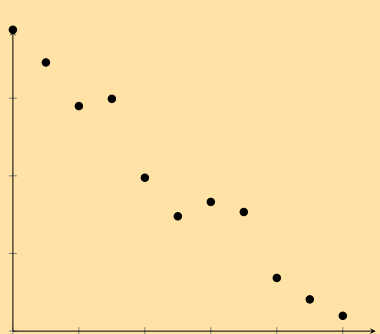
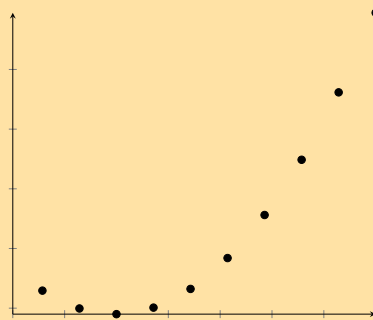
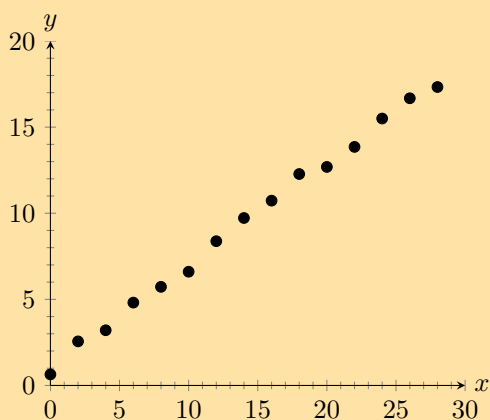
For $y = 6$:

$$\begin{aligned} 6 &= \frac{2}{3} \cdot x + 1{,}5 \\ \therefore x &= (6 - 1{,}5) \times \frac{3}{2} \\ &= 6{,}75 \end{aligned}$$

Since this y-value is outside the data range, this is **extrapolation**.

Exercise 9 – 2: Intuitive curve fitting

1. Identify the function (linear, exponential or quadratic) which would best fit the data in each of the scatter plots below:

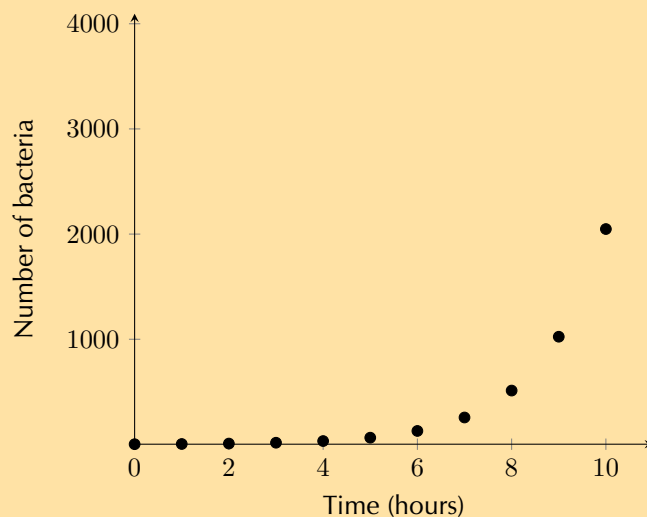
a)

d)

b)

e)

c)

f)

2. Given the scatter plot below, answer the questions that follow on the next page.

a) What type of function fits the data best? Comment on the fit of the function in terms of strength and direction.

b) Draw a line of best fit through the data and determine the equation for your line.

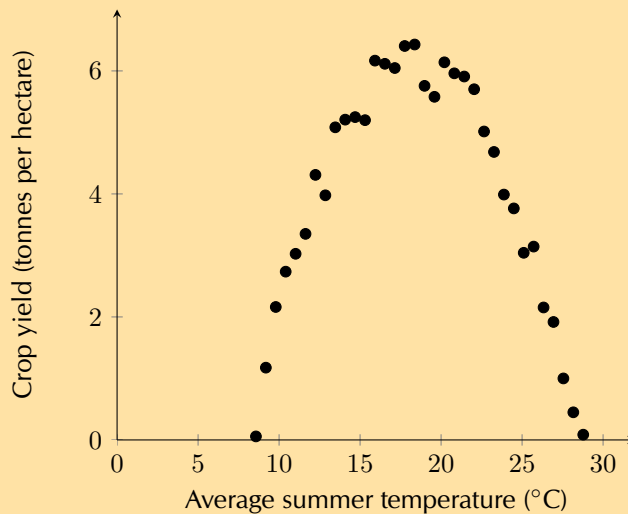
c) Using your equation, determine the estimated y-value where $x = 25$.

d) Using your equation, determine the estimated x-value where $y = 25$.

3. Tuberculosis (TB) is a disease of the lungs caused by bacteria which are spread through the air when an infected person coughs or sneezes. Drug-resistant TB arises when patients do not take their medication properly. Andile is a scientist studying a new treatment for drug-resistant TB.

   For his research, he needs to grow the TB bacterium. He takes two bacteria and puts them on a plate with nutrients for their growth. He monitors how the number of bacteria increases over time. Look at his data in the scatter plot below and answer the questions that follow.

   a) What type of function do you think fits the data best?

   b) The equation for bacterial growth is $x_n = x_0(1 + r)^t$ where x_0 is the initial number of bacteria, r is the growth rate per unit time as a proportion of 1, t is time in hours, and x_n is the number of bacteria at time, t. Determine the number of bacteria grown by Andile after 24 hours if the number of bacteria doubles every hour (i.e. the growth rate is 100% per hour).

4. Marelize is a researcher at the Department of Agriculture. She has noticed that farmers across the country have very different crop yields depending on the region. She thinks that this has to do with the different climate in each region. In order to test her idea, she collected data on crop yield and average summer temperatures from a number of farmers. Examine her data on the opposite page and answer the questions that follow.

a) Identify what type of function would fit the data best.

b) Marelize determines that the equation for the function which fits the data best is $y = -0{,}06x^2 + 2{,}2x - 14$. Determine the optimal temperature to grow wheat and the respective crop yield. Round your answer to two decimal places.

5. Dr Dandara is a scientist trying to find a cure for a disease which has an 80% mortality rate, i.e. 80% of people who get the disease will die. He knows of a plant which is used in traditional medicine to treat the disease. He extracts the active ingredient from the plant and tests different dosages (measured in milligrams) on different groups of patients. Examine his data below and complete the questions that follow.

Dosage (mg)	0	25	50	75	100	125	150	175	200
Mortality rate (%)	80	73	63	49	42	32	25	11	5

a) Draw a scatter plot of the data

b) Which function would best fit the data? Describe the fit in terms of strength and direction.

c) Draw a line of best fit through the data and determine the equation of your line.

d) Use your equation to estimate the dosage required for a 0% mortality rate.

e) Dr Dandara decided to administer the estimated dosage required for a 0% mortality rate to a group of infected patients. However, he still found a mortality rate of 5%. Name the statistical technique Dr Dandara used to estimate a mortality rate of 0% and explain why his equation did not accurately predict his experimental results.

6. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

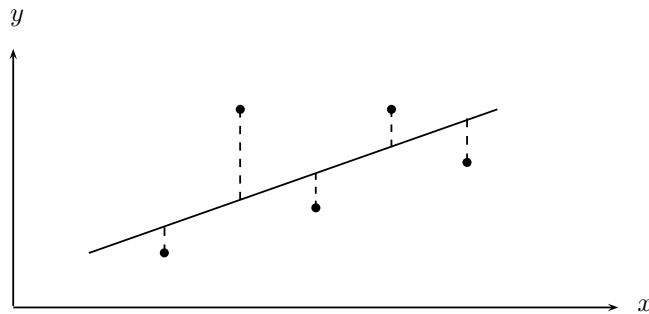
Check answers online with the exercise code below or click on 'show me the answer'.

1a. 29CS 1b. 29CT 1c. 29CV 1d. 29CW 1e. 29CX 1f. 29CY
2. 29CZ 3. 29D2 4. 29D3 5. 29D4

In the previous worked example and exercises, you drew the line of best fit by hand. This can give us a reasonable approximation of which function best fits the data when the data points are close together. However, you and your classmates may have found that you obtained slightly different answers from one another. In the next section, we will learn about a more precise way of fitting a linear function to data.

Linear regression EMCJR

Linear regression analysis is a statistical technique for finding out exactly which linear function best fits a given set of data. We can find out the equation of the regression line by using an algebraic method called the **least squares method**, available on most scientific calculators. The linear regression equation is written $\hat{y} = a + bx$ (we say y-hat) or $y = A + Bx$. Of course these are both variations of the more familiar equation $y = mx + c$.

The least squares method is very simple. Suppose we guess a line of best fit, then at every data point, we find the distance between the data point and the line. If the line fitted the data perfectly, this distance would be zero for all the data points. The worse the fit, the larger the differences. We then square each of these distances, and add them all together.

The best-fit line is then the line that minimises the sum of the squared distances.

See video: 29D5 at www.everythingmaths.co.za

Suppose we have a data set of n points $\{(x_1; y_1), (x_2; y_2), ..., (x_n; y_n)\}$. We also have a line $f(x) = mx + c$ that we are trying to fit to the data. The distance between the first data point and the line, for example, is distance $= y_1 - f(x_1) = y_1 - (mx_1 + c)$

We now square each of these distances and add them together. Lets call this sum $S(m, c)$. Then we have that

$$\begin{aligned} S(m,c) &= (y_1 - f(x_1))^2 + (y_2 - f(x_2))^2 + \ldots + (y_n - f(x_n))^2 \\ &= \sum_{i=1}^{n} (y_i - f(x_i))^2 \end{aligned}$$

Thus our problem is to find the value of m and c such that $S(m, c)$ is minimised. Let us call these minimising values b and a respectively. Then the line of best-fit is $f(x) = a + bx$. We can find a and b using calculus, but it is tricky, and we will just give you the result, which is that

$$\begin{aligned} b &= \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n} (x_i)^2 - \left(\sum_{i=1}^{n} x_i\right)^2} \\ a &= \frac{1}{n}\sum_{i=1}^{n} y_i - \frac{b}{n}\sum_{i=1}^{n} x_i = \bar{y} - b\bar{x} \end{aligned}$$

Worked example 6: Method of least squares by hand

QUESTION

In the table below, we have the records of the maintenance costs in rands compared with the age of the appliance in months. We have data for five appliances. Determine the equation for the least squares regression line by hand.

Appliance	1	2	3	4	5
Age (x)	5	10	15	20	30
Cost (y)	90	140	250	300	380

SOLUTION

Appliance	x	y	xy	x^2
1	5	90	450	25
2	10	140	1400	100
3	15	250	3750	225
4	20	300	6000	400
5	30	380	11 400	900
Total	80	1160	23 000	1650

$$
\begin{aligned}
b &= \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - \left(\sum x\right)^2} = \frac{5 \times 23000 - 80 \times 1160}{5 \times 1650 - 80^2} \\
&= 12 \\
a &= \bar{y} - b\bar{x} = \frac{1160}{5} - \frac{12 \times 80}{5} \\
&= 40 \\
\therefore \hat{y} &= 40 + 12x
\end{aligned}
$$

Worked example 7: Using the SHARP EL-531VH calculator

QUESTION

Using a calculator, find the equation of the least squares regression line for the following data:

Days (x)	1	2	3	4	5
Growth in m (y)	1,00	2,50	2,75	3,00	3,50

NB. If you have a CASIO calculator, do the next worked example first. Come back to this worked example once you are done and see if you get the same answer on your calculator.

SOLUTION

Step 1: Getting your calculator ready

Using your calculator, change the mode from normal to "Stat xy ". Do this by pressing [2ndF] and then 2. This mode enables you to type in bivariate data.

Step 2: Entering the data

Key in the data row by row:

Enter:	Press:	Enter:	Press:	See:
1	(x, y)	1	**DATA**	n = 1
2	(x, y)	2,5	**DATA**	n = 2
3	(x, y)	2,75	**DATA**	n = 3
4	(x, y)	3,0	**DATA**	n = 4
5	(x, y)	3,5	**DATA**	n = 5

Note: The [(x, y)] button is the same as the [**STO**] button and the [**DATA**] button is the same as the [M+] button.

Step 3: Getting regression results from the calculator

Ask for the values of the regression coefficients a and b.

Press:	Press:	See:
RCL	a	$a = 0{,}9$
RCL	b	$b = 0{,}55$

$\therefore \hat{y} = 0{,}9 + 0{,}55x$

Worked example 8: Using the CASIO fx-82ZA PLUS calculator

QUESTION

Using a calculator determine the least squares line of best fit for the following data set.

Learner	1	2	3	4	5
Chemistry (%)	52	55	86	71	45
Accounting (%)	48	64	95	79	50

For a Chemistry mark of 65%, what mark does the least squares line predict for Accounting?

NB. If you have a SHARP calculator, ensure that you have done the previous worked example first. Once you have completed the previous worked example, attempt this example using your calculator and see if you get the same answer.

SOLUTION

Step 1: Getting your calculator ready

Switch on the calculator. Press [MODE] and then select STAT by pressing [2]. The following screen will appear:

1:	$1 - VAR$	2:	$A + BX$
3:	$_{+}CX^{2}$	4:	lnX
5:	eX	6:	$A.BX$
7:	$A.XB$	8:	$1/X$

Now press [2] for linear regression. Your screen should look something like this:

	X	Y
1		
2		
3		

Step 2: Entering the data

Press [52] and then [=] to enter the first mark under x. Then enter the other values, in the same way, for the x-variable (the Chemistry marks) in the order in which they are given in the data set. Then move the cursor across and up and enter 48 under y opposite 52 in the x-column. Continue to enter the other y-values (the Accounting marks) in order so that they pair off correctly with the corresponding x-values.

	X	Y
1	52	48
2	55	64
3	86	95

Then press [AC]. The screen clears but the data remains stored.

Now press [SHIFT][1] to get the stats computations screen shown below.

1:	Type	2:	Data
3:	Sum	4:	Var
5:	Reg	6:	MinMax

Choose Regression by pressing [5].

1:	A	2:	B
3:	r	4:	$\hat{x}$
5:	$\hat{y}$		

Step 3: Getting regression results from the calculator

1. Press [1] and [=] to get the value of the y-intercept, $a = -5{,}065\ldots = -5{,}07$ (to two decimal places)

   Finally, to get the slope, use the following key sequence: [SHIFT][1][5][2][=]. The calculator gives $b = 1{,}169\ldots = 1{,}17$ (to two decimal places)

   The equation of the line of regression is thus:

   $\hat{y} = -5{,}07 + 1{,}17x$

2. Press [AC][65][SHIFT][1][5][5][=]

   This gives a (predicted) Accounting mark of $= 70{,}94 = 71\%$

Exercise 9 – 3: Least squares regression analysis

1. Determine the equation of the least-squares regression line using a table for the data sets below. Round a and b to two decimal places.

a)

x	10	4	9	11	11	6	8	18	9	13
y	1	0	6	3	9	5	9	8	7	15

b)

x	8	12	12	7	6	14	8	14	14	17
y	-5	4	3	-3	-5	-6	-2	0	-4	3

c)

x	-9	3	4	7	13	6	0	8	1	14
y	0	-12	-10	-14	-31	-32	-41	-52	-51	-63

2. Use your calculator to determine the equation of the least squares regression line for the following sets of data:

a)

x	0,16	0,32	3	2,6	6,12	7,68	6,16	8,56	11,24	11,96
y	5,48	10,56	13,4	15,96	15,44	16,6	17,2	22,28	22,04	24,32

b)

x	$-3{,}5$	5,5	4	1	5,5	5	3,5	5,5	7,5	8,5
y	-10	$-20{,}5$	$-30{,}5$	-46	$-46{,}5$	$-64{,}5$	-67	$-76{,}5$	$-83{,}5$	-94

c)

x	2,5	4,5	-2	9	8,5	10	7,5	3	8	15
y	-2	6	11	11,5	17	21	21	30,5	32,5	33,5

d)

x	7,24	8,24	5,34	1,66	0,32	11,46	9,34	14,24	12,9	12,34
y	$-3{,}2$	$-18{,}78$	$-21{,}1$	-32	$-31{,}2$	$-53{,}02$	-53	$-65{,}46$	$-74{,}8$	$-80{,}24$

e)

x	$-0{,}28$	2,32	0,12	4,64	3,08	7,92	5,08	8,96	10,28	7,12
y	$-6{,}88$	$-0{,}32$	3,68	4,8	11,68	19,2	20,96	24,96	29,28	33,28

f)

x	1	1,1	4,8	3,55	2,75	1,95	6,1	8,9	10,35	9,55
y	$-8{,}45$	$-5{,}95$	$-4{,}35$	0,85	$-2{,}95$	$-1{,}8$	0,25	0,05	4,8	$-3{,}05$

g)

x	1,9	1,1	−1,5	1,3	0,95	8,25	10,6	6,2	8,1	8,65
y	7	8,45	0,9	0,1	2,45	4,35	2,2	1,4	0,15	2,05

h)

x	−81,8	73,1	84	92,2	−69,7	−56,1	8,8	80,9	68,4	−40,4
y	10,6	16,1	3,6	4,6	11,9	18,3	16,6	17,6	17,7	24,1

i)

x	2,8	7,4	−2,4	4	11,3	6,9	2,5	1,7	5,4	8,2
y	12,4	13,4	15,3	15,4	16,4	19,2	21,1	19,4	21,3	25

j)

x	5	1,2	8	6	7,4	7,4	6,7	8,7	12,2	14,3
y	−4,2	−13,7	−23,7	−33,5	−43,8	−54,2	−63,9	−73,9	−84,5	−93,5

3. Determine the equation of the least squares regression line given each set of data values below. Round a and b to two decimal places in your final answer.

   a) $n = 10;\ \sum x = 74;\ \sum y = 424;\ \sum xy = 4114{,}51;\ \sum(x^2) = 718{,}86$

   b) $n = 13;\ \bar{x} = 8{,}45;\ \bar{y} = 17{,}83;\ \sum xy = 1879{,}25;\ \sum(x^2) = 855{,}45$

   c) $n = 10;\ \bar{x} = 5{,}77;\ \bar{y} = 17{,}03;\ \overline{xy} = 133{,}817;\ \sigma_x = \pm 3{,}91$
   (Hint: multiply the numerator and denominator of the formula for b by $\frac{1}{n^2}$)

4. The table below shows the average maintenance cost in rands of a certain model of car compared to the age of the car in years.

Age (x)	1	3	5	6	8	9	10
Cost (y)	1000	1500	1600	1800	2000	2400	2600

   a) Draw a scatter plot of the data.

   b) Complete the table below, filling in the totals of each column in the final row:

Age (x)	**Cost** (y)	xy	x^2
1	1000		
3	1500		
5	1600		
6	1800		
8	2000		
9	2400		
10	2600		
$\sum = \ldots$	$\sum = \ldots$	$\sum = \ldots$	$\sum = \ldots$

   c) Use your table to determine the equation of the least squares regression line. Round a and b to two decimal places.

   d) Use your equation to estimate what it would cost to maintain this model of car in its 15^{th} year.

   e) Use your equation to estimate the age of the car in the year where the maintenance cost totals over R 3000 for the first time.

5. Miss Colly has always maintained that there is a relationship between a learner's ability to understand the language of instruction and their marks in Mathematics. Since she teaches Mathematics through the medium of English, she decides to compare the Mathematics and English marks of her learners in order to investigate the relationship between the two marks. Examine her data below and answer the quesitons on the following page.

English % (x)	28	33	30	45	45	55	55	65	70	76	65	85	90
Mathematics % (y)	35	36	34	45	50	40	60	50	65	85	70	80	90

a) Complete the table below, filling in the totals of each column in the final row:

English % (x)	**Mathematics % (y)**	xy	x^2
28	35		
33	36		
30	34		
45	45		
45	50		
55	40		
65	50		
70	65		
76	85		
65	70		
85	80		
90	90		
$\sum = \ldots$	$\sum = \ldots$	$\sum = \ldots$	$\sum = \ldots$

b) Use your table to determine the equation of the least squares regression line. Round a and b to two decimal places.

c) Use your equation to estimate the Mathematics mark of a learner who obtained 50% for English, correct to two decimal places.

d) Use your equation to estimate the English mark of a learner who obtained 75% for Mathematics, correct to two decimal places.

6. Foot lengths and heights of ten students are given in the table below.

Height (cm)	170	163	131	181	146	134	166	172	185	153
Foot length (cm)	27	23	20	28	22	20	24	26	29	22

a) Using foot length as your x-variable, draw a scatter plot of the data.

b) Identify and describe any trends shown in the scatter plot.

c) Find the equation of the least squares line using the formulae and draw the line on your graph. Round a and b to two decimal places in your final answer.

d) Confirm your calculations above by finding the least squares regression line using a calculator.

e) Use your equation to predict the height of a student with a foot length of $21{,}6$ cm.

f) Use your equation to predict the foot length of a student 190 cm tall, correct to two decimal places.

7. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 29D6 1b. 29D7 1c. 29D8 2a. 29D9 2b. 29DB 2c. 29DC
2d. 29DD 2e. 29DF 2f. 29DG 2g. 29DH 2h. 29DJ 2i. 29DK
2j. 29DM 3a. 29DN 3b. 29DP 3c. 29DQ 4. 29DR 5. 29DS
6. 29DT

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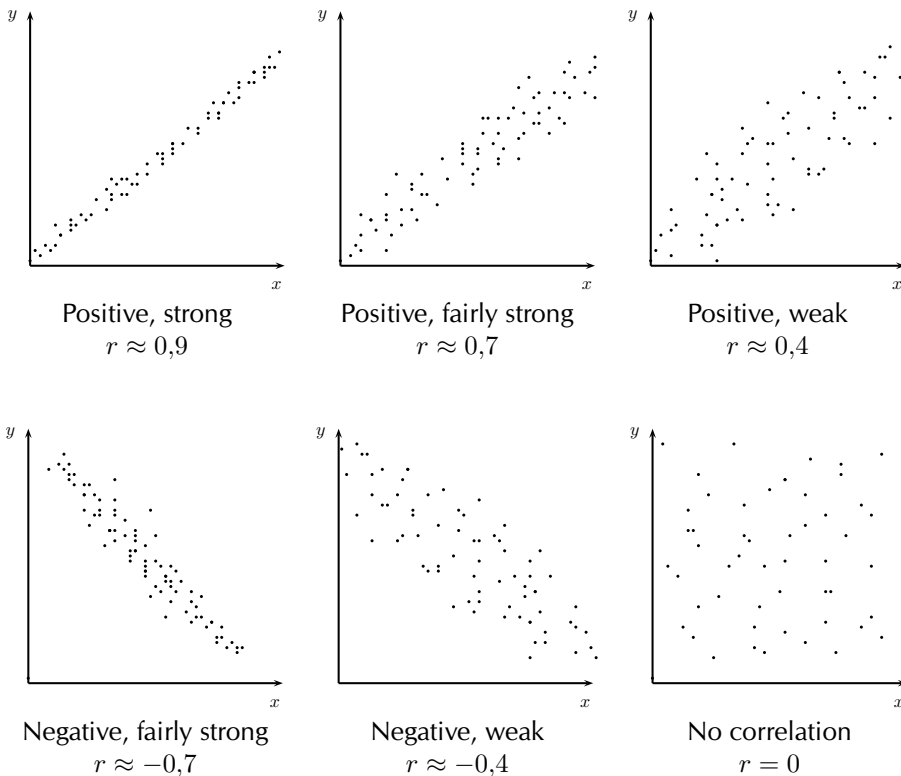
Now that we have a precise technique for finding the line of best fit, we still do not know how well our line of best fit really fits our data. We can fit a least squares regression line to any bivariate data, even if the two variables do not show a linear relationship. If the fit is not "good", our assumption of the a and b values in $\hat{y} = a + bx$ might be incorrect. Next, we will learn of a quantitative measure to determine how well our line really fits our data.

9.3 Correlation EMCJS

The linear correlation coefficient, r, is a measure which tells us the strength and direction of a relationship between two variables. The correlation coefficient $r \in [-1; 1]$. When $r = -1$, there is perfect negative correlation, when $r = 0$, there is no correlation and when $r = 1$ there is perfect positive correlation.

The linear correlation coefficient r can be calculated using the formula $r = b\frac{\sigma_x}{\sigma_y}$

- where b is the gradient of the least squares regression line,
- σ_x is the standard deviation of the x-values and
- σ_y is the standard deviation of the y-values.

This is known as the Pearson's product moment correlation coefficient. It is much easier to do on a calculator where you simply follow the procedure to calculate the regression equation, and go on to find r.

In general:

Positive	Strength	Negative
$r = 0$	no correlation	$r = 0$
$0 < r < 0{,}25$	very weak	$-0{,}25 < r < 0$
$0{,}25 < r < 0{,}5$	weak	$-0{,}5 < r < -0{,}25$
$0{,}5 < r < 0{,}75$	moderate	$-0{,}75 < r < -0{,}5$
$0{,}75 < r < 0{,}9$	strong	$-0{,}9 < r < -0{,}75$
$0{,}9 < r < 1$	very strong	$-1 < r < -0.9$
$r = 1$	perfect correlation	$r = -1$

NOTE:
Correlation does not imply causation! Just because two variables are correlated does not mean that they are causally linked, i.e. if A and B are correlated, that does not mean A causes B, or vice versa. This is a common mistake made by many people, especially journalists looking for their next juicy story.

For example, ice cream sales and shark attacks are correlated. This does not mean that the sale of ice cream is somehow causing more shark attacks. Instead, a simpler explanation is that the warmer it is, the more likely people are to buy ice cream and the more likely people are to go to the beach as well, thus increasing the likelihood of a shark attack.

See video: 29DV at www.everythingmaths.co.za

Worked example 9: The correlation coefficient

QUESTION

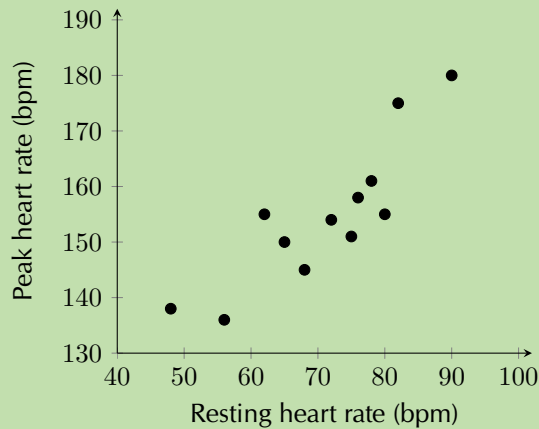
A cardiologist wanted to test the relationship between resting heart rate and the peak heart rate during exercise. Heart rate is measured in beats per minute (bpm). The following set of data was generated from 12 study participants after they had run on a treadmill at 10 km/h for 10 minutes.

Resting heart rate	48	56	90	65	75	78	80	72	82	76	68	62
Peak heart rate	138	136	180	150	151	161	155	154	175	158	145	155

- Draw a scatter plot of the data. Use resting heart rate as your x-variable.
- Use your calculator to determine the equation of the line of best fit.
- Estimate what the heart rate of a person with a resting heart rate of 70 bpm will be after exercise.
- Without using your calculator, find the correlation coefficient, r. Confirm your answer using your calculator.
- What can you conclude regarding the relationship between resting heart rate and the heart rate after exercise?

SOLUTION

Step 1: Draw the scatter plot

1. Choose a suitable scale for the axes.
2. Draw the axes.
3. Plot the points.

Step 2: Calculate the equation of the line of best fit
As you learnt previously, use your calculator to determine the values for a and b.

$a = 86{,}75$

$b = 0{,}96$

Therefore, the equation for the line of best fit is $y = 86{,}75 + 0{,}96x$

Step 3: Calculate the estimated value for y
If $x = 70$, using our equation, the estimated value for y is:

$$y = 86{,}75 + 0{,}96 \times 70 = 153{,}95$$

Step 4: Calculate the correlation co-efficient
The formula for r is:

$$r = b\frac{\sigma_x}{\sigma_y}$$

We already know the value of b and you know how to calculate b by hand from worked example 5, so we are just left to determine the value for σ_x and σ_y. The formula for standard deviation is:

$$\sigma_x = \frac{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}}{n}$$

First, you need to determine $\bar{x}$ and $\bar{y}$ and then complete a table like the one below.

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} = 71$$

$$\bar{y} = \frac{\sum_{i=1}^{n} y_i}{n} = 154{,}83 \text{ (rounded to two decimal places)}$$

Resting heart rate (x)	**Peak heart rate (y)**	$(x-\bar{x})^2$	$(y-\bar{y})^2$
48	138	529	283,25
56	136	225	354,57
90	180	361	633,53
65	150	36	23,33
75	151	16	14,67
78	161	49	38,07
80	155	81	0,03
72	154	1	0,69
82	175	121	406,83
76	158	25	10,05
68	145	9s	96,63
62	155	81	0,03
$\sum = 852$	$\sum = 1858$	$\sum = 1534$	$\sum = 1861{,}68$

$$\sigma_x = \frac{\sqrt{\sum_{i=1}^{n}(x_i-\bar{x})^2}}{n} = \frac{\sqrt{1534}}{12} = \pm 3{,}26$$

$$\sigma_y = \frac{\sqrt{\sum_{i=1}^{n}(y_i-\bar{y})^2}}{n} = \frac{\sqrt{1861{,}68}}{12} = \pm 3{,}60$$

$$b = 0{,}96$$

$$\therefore r = 0{,}96 \times \frac{3{,}26}{3{,}60}$$

$$= 0{,}87$$

Step 5: Confirm your answer using your calculator

Once you know the method for finding the equation of the best line of fit on your calculator, finding the value for r is trivial. After you have entered all your x and y values into your calculator, in STAT mode:

- on a SHARP calculator: press [RCL] then [r] (the same key as [÷])
- on a CASIO calculator: press [SHIFT] then [STAT], [5], [3] then [=]

Step 6: Comment on the correlation coefficient

$$r = 0{,}87$$

Therefore, there is a strong, positive, linear relationship between resting heart rate and peak heart rate during exercise. This means that the higher your resting heart rate, the higher your peak heart rate during exercise is likely to be.

Exercise 9 – 4: Correlation coefficient

1. Determine the correlation coefficient by hand for the following data sets and comment on the strength and direction of the correlation. Round your answers to two decimal places.

a)

x	5	8	13	10	14	15	17	12	18	13
y	5	8	3	8	7	5	3	-1	4	-1

b)

x	7	3	11	7	7	6	9	12	10	15
y	13	23	32	45	50	55	67	69	85	90

c)

x	3	10	7	6	11	16	17	15	17	20
y	6	24	30	38	53	56	65	75	91	103

2. Using your calculator, determine the value of the correlation coefficient to two decimal places for the following data sets and describe the strength and direction of the correlation.

a)

x	0,1	0,8	1,2	3,4	6,5	3,9	6,4	7,4	9,9	8,5
y	$-5,1$	-10	$-17,3$	$-24,9$	$-31,9$	$-38,6$	-42	-55	-62	$-64,8$

b)

x	-26	-34	-51	-14	50	-57	-11	-10	36	-35
y	-66	-10	-26	-51	-58	-56	45	-142	-149	-30

c)

x	101	-398	103	204	105	606	807	-992	609	-790
y	-300	98	-704	-906	-8	690	-12	686	984	-18

d)

x	101	82	-7	-6	45	-94	-23	78	-11	0
y	111	-74	21	106	51	26	21	86	-29	66

e)

x	-3	5	-4	0	-2	9	10	11	17	9
y	24	18	21	30	31	39	48	59	56	54

3. Calculate and describe the direction and strength of r for each of the sets of data values below. Round all r-values to two decimal places.

a) $b = -1{,}88; \ \sigma_x^2 = 48{,}62; \ \sigma_y^2 = 736{,}54.$

b) $a = 32{,}19; \ x = 4{,}3; \ \bar{y} = 36{,}6; \ \sum_{i=1}^{n}(x_i - \bar{x})^2 = 620{,}1; \ \sum_{i=1}^{n}(y_i - \bar{y})^2 = 2636{,}4.$

4. The geography teacher, Mr Chadwick, gave the data set below to his class to illustrate the concept that average temperature depends on how far a place is from the equator (known as the latitude). There are 90 degrees between the equator and the North Pole. The equator is defined as 0 degrees. Examine the data set below and answer the questions that follow.

City	**Degrees N** (x)	**Ave. temp.** (y)	xy	x^2	$(x-\bar{x})^2$	$(y-\bar{y})^2$
Cairo	43	22				
Berlin	53	19				
London	40	18				
Lagos	6	32				
Jerusalem	31	23				
Madrid	40	28				
Brussels	51	18				
Istanbul	39	23				
Boston	43	23				
Montreal	45	22				
Total:						

a) Copy and complete the table.

b) Using your table, determine the equation of the least squares regression line. Round a and b to two decimal places in your final answer.

c) Use your calculator to confirm your equation for the least squares regression line.

d) Using your table, determine the value of the correlation coefficient to two decimal places.

e) What can you deduce about the relationship between how far north a city is and its average temperature?

f) Estimate the latitude of Paris if it has an average temperature of $25°$C

5. A taxi driver recorded the number of kilometres his taxi travelled per trip and his fuel cost per kilometre in Rands. Examine the table of his data below and answer the questions that follow.

Distance (x)	3	5	7	9	11	13	15	17	20	25	30
Cost (y)	2,8	2,5	2,46	2,42	2,4	2,36	2,32	2,3	2,25	2,22	2

a) Draw a scatter plot of the data.

b) Use your calculator to determine the equation of the least squares regression line and draw this line on your scatter plot. Round a and b to two decimal places in your final answer.

c) Using your calculator, determine the correlation coefficient to two decimal places.

d) Describe the relationship between the distance travelled per trip and the fuel cost per kilometre.

e) Predict the distance travelled if the cost per kilometre is R $1{,}75$.

6. The time taken, in seconds, to complete a task and the number of errors made on the task were recorded for a sample of 10 primary school learners. The data is shown in the table below. [Adapted from NSC Paper 3 Feb-March 2013]

Time taken to complete task (in seconds)	23	21	19	9	15	22	17	14	21	18
Number of errors made	2	4	5	9	7	3	7	8	3	5

   a) Draw a scatter plot of the data.
   b) What is the influence of more time taken to complete the task on the number of errors made?
   c) Determine the equation of the least squares regression line and draw this line on your scatter plot. Round a and b to two decimal places in your final answer.
   d) Determine the correlation coefficient to two decimal places.
   e) Predict the number of errors that will be made by a learner who takes 13 seconds to complete this task.
   f) Comment on the strength of the relationship between the variables.

7. A recording company investigates the relationship between the number of times a CD is played by a national radio station and the national sales of the same CD in the following week. The data below was collected for a random sample of 10 CDs. The sales figures are rounded to the nearest 50. [NSC Paper 3 November 2012]

Number of times CD is played	47	34	40	34	33	50	28	53	25	46
Weekly sales of the CD	3950	2500	3700	2800	2900	3750	2300	4400	2200	3400

   a) Draw a scatter plot of the data.
   b) Determine the equation of the least squares regression line.
   c) Calculate the correlation coefficient.
   d) Predict, correct to the nearest 50, the weekly sales for a CD that was played 45 times by the radio station in the previous week.
   e) Comment on the strength of the relationship between the variables.

8. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 29DW 1b. 29DX 1c. 29DY 2a. 29DZ 2b. 29F2 2c. 29F3
2d. 29F4 2e. 29F5 3a. 29F6 3b. 29F7 4. 29F8 5. 29F9
6. 29FB 7. 29FC

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9.4 Summary EMCJT

- **Curve fitting** is the process of fitting functions to data.
- **Intuitive** curve fitting is performed by visually interpreting if the points on the scatter plot conform to a linear, exponential, quadratic or some other function.
- The **line of best fit** or trend line is a straight line through the data which best approximates the available data points. This allows for the estimation of missing data values.
- **Interpolation** is the technique used to predict values that fall within the range of the available data.
- **Extrapolation** is the technique used to predict the value of variables beyond the range of the available data.
- **Linear regression analysis** is a statistical technique of finding out exactly which linear function best fits a given set of data.
- The **least squares method** is an algebraic method of finding the linear regression equation. The linear regression equation is written $\hat{y} = a + bx$, where

$$b = \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n} (x_i)^2 - \left(\sum_{i=1}^{n} x_i\right)^2}$$

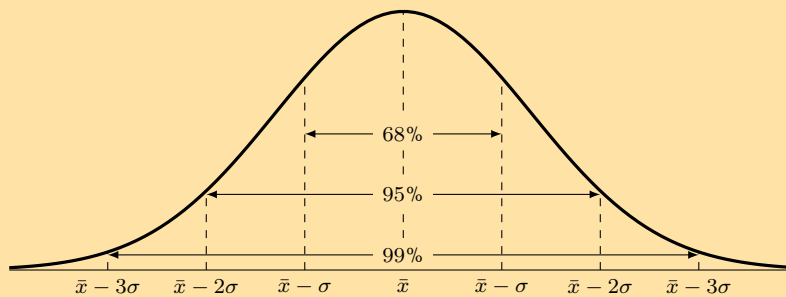
$$a = \frac{1}{n}\sum_{i=1}^{n} y_i - \frac{b}{n}\sum_{i=1}^{n} x_i = \bar{y} - b\bar{x}$$

- The **linear correlation coefficient**, r, is a measure which tells us the strength and direction of a relationship between two variables, determined using the equation:

$$r = b\left(\frac{\sigma_x}{\sigma_y}\right)$$

- The correlation coefficient $r \in [-1; 1]$. When $r = -1$, there is perfect negative correlation, when $r = 0$, there is no correlation and when $r = 1$, there is perfect positive correlation.

Exercise 9 – 5: End of chapter exercises

1. The number of SMS messages sent by a group of teenagers was recorded over a period of a week. The data was found to be normally distributed with a mean of 140 messages and a standard deviation of 12 messages. [NSC Paper 3 Feb-March 2012]

Answer the following questions with reference to the information provided in the graph:

a) What percentage of teenagers sent less than 128 messages?

b) What percentage of teenagers sent between 116 and 152 messages?

2. A company produces sweets using a machine which runs for a few hours per day. The number of hours running the machine and the number of sweets produced are recorded.

Machine hours	Sweets produced
3,80	275
4,23	287
4,37	291
4,10	281
4,17	286

Find the linear regression equation for the data, and estimate the machine hours needed to make 300 sweets.

3. The profits of a new shop are recorded over the first 6 months. The owner wants to predict his future sales. The profits by month so far have been R 90 000; R 93 000; R 99 500; R 102 000; R 101 300; R 109 000.

a) Calculate the linear regression function for the data, using profit as your y-variable. Round a and b to two decimal places.

b) Give an estimate of the profits for the next two months.

c) The owner wants a profit of R 130 000. Estimate how many months this will take.

4. A fast food company produces hamburgers. The number of hamburgers made and the costs are recorded over a week. Examine the data below and answer the questions on the following page.

Hamburgers made	Costs
495	R 2382
550	R 2442
515	R 2484
500	R 2400
480	R 2370
530	R 2448
585	R 2805

a) Find the linear regression function that best fits the data. Use hamburgers made as your x-variable and round a and b to two decimal places.

b) Calculate the value of the correlation coefficient, correct to two decimal places, and comment on the strength and direction of the correlation.

c) If the total cost in a day is R 2500, estimate the number of hamburgers produced. Round your answer down to the nearest whole number.

d) What is the cost of 490 hamburgers?

5. A collection of data related to an investigation into biceps length and height of students was recorded in the table below. Answer the questions to follow.

Length of right biceps (cm)	Height (cm)
25,5	163,3
26,1	164,9
23,7	165,5
26,4	173,7
27,5	174,4
24	156
22,6	155,3
27,1	169,3

a) Draw a scatter plot of the data set.

b) Calculate equation of the line of regression.

c) Draw the regression line onto the graph.

d) Calculate the correlation coefficient r

e) What conclusion can you reach, regarding the relationship between the length of the right biceps and height of the students in the data set?

6. A class wrote two tests, and the marks for each were recorded in the table below. Full marks in the first test was 50, and the second test was out of 30.

Learner	Test 1	Test 2
	(Full marks: 50)	(Full marks: 30)
1	42	25
2	32	19
3	31	20
4	42	26
5	35	23
6	23	14
7	43	24
8	23	12
9	24	14
10	15	10
11	19	11
12	13	10
13	36	22
14	29	17
15	29	17
16	25	16
17	29	18
18	17	
19	30	19
20	28	17

a) Is there a strong correlation between the marks for the first and second test? Show why or why not.

b) One of the learners (in Row 18) did not write the second test. Given her mark for the first test, calculate an expected mark for the second test. Round the mark up to the nearest whole number.

7. Lindiwe works for Eskom, the South African power distributor. She knows that on hot days more electricity than average is used to cool houses. In order to accurately predict how much more electricity needs to be produced, she wants to determine the precise nature of the relationship between temperature and electricity usage.

The data below shows the peak temperature in degrees Celsius on ten consecutive days during summer and the average number of units of electricity used by a number of households. Examine her data and answer the questions that follow.

Peak temp. (y)	32	40	30	28	25	38	36	20	24	26
Ave. no. of units (x)	37	45	35	30	20	40	38	15	20	22

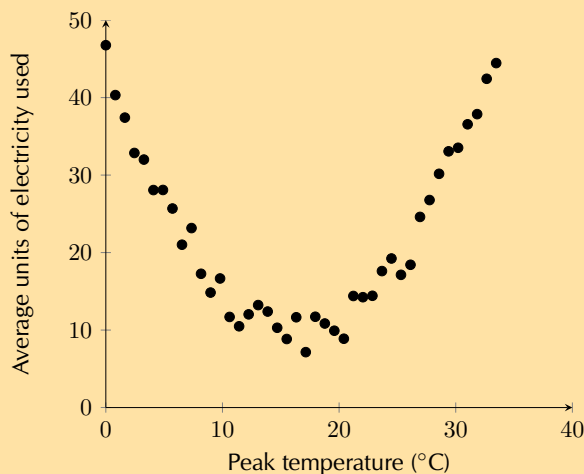
a) Draw a scatter plot of the data.

b) Using the formulae for a and b, determine the equation of the least squares line.

c) Determine the value of the correlation coefficient, r, by hand.

d) What can Lindiwe conclude about the relationship between peak temperature and the number of electricity units used?

e) Predict the average number of units of electricity used by a household on a day with a peak temperature of 45°C. Give your answer correct to the nearest unit and identify what this type of prediction is called.

f) Lindiwe suspected that the relationship between temperature and electricity consumption was not linear for all temperatures. She then decided to collect data for peak temperatures down to 0°C. Examine the graph of her data below and identify which type of function would best fit the data and describe the nature of the relationship between temperature and electricity for the newly available data.

g) Lindiwe is asked by her superiors to determine which day is best to perform maintenance on one of their power plants. She determined that the equation $y = 0{,}13x^2 - 4{,}3x + 45$ best fit her data. Use her equation to estimate the peak temperature and average no. of units used on the day when the least amount of electricity generation is required.

8. Below is a list of data concerning 12 countries and their respective carbon dioxide (CO_2) emission levels per person per annum (measured in tonnes) and the gross domestic product (GDP is a measure of products produced and services delivered within a country in a year) per person (in US dollars). Data sourced from the World Bank and the US Department of Energy's Carbon Dioxide Information Analysis Center.

	CO_2 emmissions per capita (x)	GDP per capita (y)
South Africa	8,8	11 440
Thailand	4,1	9815
Italy	7,5	32 512
Australia	18,3	44 462
China	5,3	9233
India	1,4	3876
Canada	15,3	42 693
United Kingdom	8,5	35 819
United States	17,2	49 965
Saudi Arabia	16,1	24 571
Iran	7,3	11 395
Indonesia	1,8	4956

a) Draw a scatter plot of the data set.

b) Draw your estimate of the line of best fit on your scatter plot and determine the equation of your line of best fit.

c) Use your calculator to determine the equation for the least squares regression line. Round a and b to two decimal places in your final answer.

d) Use your calculator to determine the correlation coefficient, r. Round your answer to two decimal places.

e) What conclusion can you reach regarding the relationship between CO_2 emissions per annum and GDP per capita for the countries in the data set?

f) Kenya has a GDP per capita of \$1712. Use your equation of the least squares regression line to estimate the annual CO_2 emissions of Kenya correct to two decimal places.

9. A group of students attended a course in Statistics on Saturdays over a period of 10 months. The number of Saturdays on which a student was absent was recorded against the final mark the student obtained. The information is shown in the table below. [Adapted from NSC Paper 3 Feb-March 2012]

Number of Saturdays absent	0	1	2	2	3	3	5	6	7
Final mark (as %)	96	91	78	83	75	62	70	68	56

a) Draw a scatter plot of the data.

b) Determine the equation of the least squares line and draw it on your scatter plot.

c) Calculate the correlation coefficient.

d) Comment on the trend of the data.

e) Predict the final mark of a student who was absent for four Saturdays.

10. Grant and Christie are training for a half-marathon together in 8 weeks time. Christie is much fitter than Grant but she has challenged him to beat her time at the race. Grant has begun a rigid training programme to try and improve his time.

    Time taken to complete a half marathon was recorded each Sunday. The first recorded Sunday is denoted as week 1. The half-marathon takes place on the eighth Sunday, i.e. week 8. Examine the data set in the table below and answer the questions the follow.

Week	1	2	3	4	5	6
Grant's time (HH:MM)	02:01	01:59	01:55	01:53	01:47	01:42
Christie's time (HH:MM)	01:40	01:42	01:38	01:39	01:37	01:35

    a) Draw a scatter plot of the data sets. Include Grant and Christie's data on the same set of axes. Use a • to denote Grant's data points and $\times$ to denote Christie's data points. Convert all times to minutes.
    b) Comment on and compare any trends that you observe in the data.
    c) Determine the equations of the least squares regression lines for Grant's data and Christie's data. Draw these lines on your scatter plot. Use a different colour for each.
    d) Calculate the correlation coefficient and comment on the fit for each data set.
    e) Assuming the observed trends continue, will Grant beat Christie at the race?
    f) Assuming the observed trends continue, extrapolate the week in which Grant will be able to run a half-marathon in less time than Christie.

11. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 29FD 2. 29FF 3. 29FG 4. 29FH 5. 29FJ 6. 29FK
7. 29FM 8. 29FN 9. 29FP 10. 29FQ

CHAPTER

Probability

10.1 Revision EMCJV

Terminology EMCJW

Outcome: a single observation of an uncertain or random process (called an *experiment*). For example, when you accidentally drop a book, it might fall on its cover, on its back or on its side. Each of these options is a possible outcome.

Sample space of an experiment: the set of all possible outcomes of the experiment. For example, the sample space when you roll a single 6-sided die is the set $\{1; 2; 3; 4; 5; 6\}$. For a given experiment, there is exactly one sample space. The sample space is denoted by the letter S.

Event: a set of outcomes of an experiment. For example, if you have a standard deck of 52 cards, an event may be picking a spade card or a king card.

Probability of an event: a real number between and inclusive of 0 and 1 that describes how likely it is that the event will occur. A probability of 0 means the outcome of the experiment will never be in the event set. A probability of 1 means the outcome of the experiment will always be in the event set. When all possible outcomes of an experiment have equal chance of occurring, the probability of an event is the number of outcomes in the event set as a fraction of the number of outcomes in the sample space. To calculate a probability, you divide the number of favourable outcomes by the total number of possible outcomes.

Relative frequency of an event: the number of times that the event occurs during experimental trials, divided by the total number of trials conducted. For example, if we flip a coin 10 times and it landed on heads 3 times, then the relative frequency of the heads event is $\frac{3}{10} = 0{,}3$.

Union of events: the set of all outcomes that occur in at least one of the events. For 2 events called A and B, we write the union as "A or B". Another way of writing the union is using set notation: $A \cup B$. For example, if A is all the countries in Africa and B is all the countries in Europe, A or B is all the countries in Africa and Europe.

Intersection of events: the set of all outcomes that occur in all of the events. For 2 events called A and B, we write the intersection as "A and B". Another way of writing the intersection is using set notation: $A \cap B$. For example, if A is soccer players and B is cricket players, A and B refers to those who play both soccer and cricket.

Mutually exclusive events: events with no outcomes in common, that is (A and B) is an empty set. Mutually exclusive events can never occur simultaneously. For example the event that a number is even and the event that the same number is odd are mutually exclusive, since a number can never be both even and odd.

Complementary events: two mutually exclusive events that together contain all the outcomes in the sample space. For an event called A, we write the complement as "not A". Another way of writing the complement is as A'.

Dependent and independent events: two events, A and B, are **independent** if the outcome of the first event does not influence the outcome of the second event. For example, if you flip a coin and it lands on tails and flip it again and it lands on heads, neither outcome influences the other. Two events, C and D, are **dependent** if the outcome of one event influences the outcome of the other. For example, if your lunchbox contains 3 sandwiches and 2 apples, when you eat one of the items, this reduces the number of choices you have when deciding to eat a second item.

See video: 29FR at www.everythingmaths.co.za

10.2 Identities EMCJX

The **addition rule** (also called the sum rule) for any 2 events, A and B is

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

This rule relates the probabilities of 2 events with the probabilities of their union and intersection.

The **addition rule for 2 mutually exclusive events** is

$$P(A \text{ or } B) = P(A) + P(B)$$

This rule is a special case of the previous rule. Because the events are mutually exclusive, $P(A \text{ and } B) = 0$.

The **complementary rule** is

$$P(\text{not } A) = 1 - P(A)$$

This rule is a special case of the previous rule. Since A and (not A) are complementary, $P(A \text{ or } (\text{not } A)) = 1$.

The **product rule** for independent events A and B is:

$$P(A \text{ and } B) = P(A) \times P(B)$$

If two events A and B are **dependent** then:

$$P(A \text{ and } B) \neq P(A) \times P(B)$$

WARNING!

Just because two events are mutually exclusive does not necessarily mean that they are independent. To test whether events are mutually exclusive, always check that $P(A \text{ and } B) = 0$. To test whether events are independent, always check that $P(A \text{ and } B) = P(A) \times P(B)$. See the exercises below for examples of events that are mutually exclusive and independent in different combinations.

Worked example 1: Dependent and independent events

QUESTION

Write down which of the following events are dependent and which are independent:

1. The student council chooses a head student and then a deputy head student.
2. A bag contains blue marbles and red marbles. You take a red marble out of the bag and then throw it back in again before you take another marble out of the bag.

SOLUTION

Step 1: Ask the question: Did the available choices change for the second event because of the first event?

1. Yes, because after selecting the head student there are fewer council members available to choose for the deputy head student position. Therefore the two events are dependent.
2. No, because when you throw the first marble back into the bag, there are the same number and colour composition of choices for the second marble. Therefore the two events are independent.

Worked example 2: Independent and dependent events

QUESTION

A bag contains 3 yellow and 4 black beads. We remove a random bead from the bag, record its colour and put it back into the bag. We then remove another random bead from the bag and record its colour.

1. What is the probability that the first bead is yellow?
2. What is the probability that the second bead is black?
3. What is the probability that the first bead is yellow and the second bead is black?
4. Are the first bead being yellow and the second bead being black independent events?

SOLUTION

Step 1: Probability of a yellow bead first

Since there is a total of 7 beads, of which 3 are yellow, the probability of getting a yellow bead is

$$P(\text{first bead yellow}) = \frac{3}{7}$$

Step 2: Probability of a black bead second

The problem states that the first bead is placed back into the bag before we take the second bead. This means that when we draw the second bead, there are again a total of 7 beads in the bag, of which 4 are black. Therefore the probability of drawing a black bead is

$$P(\text{second bead black}) = \frac{4}{7}$$

Step 3: Probability of yellow first and black second

When drawing two beads from the bag, there are 4 possibilities. We can get

- a yellow bead and then another yellow bead;
- a yellow bead and then a black bead;
- a black bead and then a yellow bead;
- a black bead and then another black bead.

We want to know the probability of the second outcome, where we have to get a yellow bead first. Since there are 3 yellow beads and 7 beads in total, there are $\frac{3}{7}$ ways to get a yellow bead first. Now we put the first bead back, so there are again 3 yellow beads and 4 black beads in the bag. Therefore there are $\frac{4}{7}$ ways to get a black bead second if the first bead was yellow. This means that there are

$$\frac{3}{7} \times \frac{4}{7} = \frac{12}{49}$$

ways to get a yellow bead first and a black bead second. So, the probability of getting a yellow bead first and a black bead second is $\frac{12}{49}$.

Step 4: Dependent or independent?

According to the definition, events are independent if and only if

$$P(A \text{ and } B) = P(A) \times P(B)$$

In this problem:

- $P(\text{first bead yellow}) = \frac{3}{7}$
- $P(\text{second bead black}) = \frac{4}{7}$
- $P(\text{first bead yellow and second bead black}) = \frac{12}{49}$

Since $\frac{12}{49} = \frac{3}{7} \times \frac{4}{7}$, the events are independent.

See video: 29FS at www.everythingmaths.co.za

Worked example 3: Independent and dependent events

QUESTION

In the previous example, we picked a random bead and put it back into the bag before continuing. This is called **sampling with replacement**. In this worked example, we will follow the same process, except that we will not put the first bead back into the bag. This is called **sampling without replacement**.

So, from a bag with 3 red and 5 green beads, we remove a random bead and record its colour. Then, without putting back the first bead, we remove another random bead from the bag and record its colour.

1. What is the probability that the first bead is red?
2. What is the probability that the second bead is green?
3. What is the probability that the first bead is red and the second bead is green?
4. Are the first bead being red and the second bead being green independent events?

SOLUTION

Step 1: Count the number of outcomes

We will examine the number ways in which we can get the different possible outcomes when removing 2 beads. The possible outcomes are

- a red bead and then another red bead (RR);
- a red bead and then a green bead (RG);
- a green bead and then a red bead (GR);
- a green bead and then another green bead (GG).

For the first outcome, we have to get a red bead first. Since there are 3 red beads and 8 beads in total, there are $\frac{3}{8}$ ways to get a red bead first. After we have taken out a red bead, there are now 2 red beads and 5 green beads left. Therefore there are $\frac{2}{7}$ ways to get a red bead second if the first bead was also red. This means that there are

$$\frac{3}{8} \times \frac{2}{7} = \frac{6}{56} = \frac{3}{28}$$

ways to get a red bead first and a red bead second. The probability of the first outcome is $\frac{3}{28}$.

For the second outcome, we have to get a red bead first. As in the first outcome, there are $\frac{3}{8}$ ways to get a red bead first; and there are now 2 red beads and 5 green beads left. Therefore there are $\frac{5}{7}$ ways to get a green bead second if the first bead was red. This means that there are

$$\frac{3}{8} \times \frac{5}{7} = \frac{15}{56}$$

ways to get a red bead first and a green bead second. The probability of the second outcome is $\frac{15}{56}$.

In the third outcome, the first bead is green and the second bead is red. There are $\frac{5}{8}$ ways to get a green bead first; and there are now 4 green beads and 3 red beads left. Therefore there are $\frac{3}{7}$ ways to get a red bead second if the first bead was green. This means that there are

$$\frac{5}{8} \times \frac{3}{7} = \frac{15}{56}$$

ways to get a red bead first and a green bead second. The probability of the third outcome is $\frac{15}{56}$.

In the fourth outcome, the first and second beads are both green. Since there are 5 green beads and 8 beads in total, there are $\frac{5}{8}$ ways to get a green bead first. After we have removed a green bead, 3 red beads and 4 green beads remain in the bag. Therefore there are $\frac{4}{7}$ ways to get a green bead second if the first bead was also green. This means that there are

$$\frac{5}{8} \times \frac{4}{7} = \frac{20}{56} = \frac{5}{14}$$

ways to get a green bead first and a green bead second. Therefore the probability of the fourth outcome is $\frac{5}{14}$.

To summarise, these are the possible outcomes and their probabilities:

- first bead red and second bead red (RR): $\frac{3}{28}$;
- first bead red and second bead green (RG): $\frac{15}{56}$;
- first bead green and second bead red (GR): $\frac{15}{56}$;
- first bead green and second bead green (GG): $\frac{5}{14}$.

Step 2: Probability of a red bead first

To determine the probability of getting a red bead on the first draw, we look at all of the outcomes that contain a red bead first. These are

- a red bead and then another red bead (RR);
- a red bead and then a green bead (RG).

The probability of the first outcome is $\frac{3}{28}$ and the probability of the second outcome is $\frac{15}{56}$. By adding these two probabilities, we see that the probability of getting a red bead first is

$$P(\text{first bead red}) = \frac{3}{28} + \frac{15}{56} = \frac{6}{56} + \frac{15}{56} = \frac{21}{56} = \frac{3}{8}$$

This is the same as in the previous worked example, which should not be too surprising since the probability of the first bead being red is not affected by whether or not we put it back into the bag before drawing the second bead.

Step 3: Probability of a green bead second

To determine the probability of getting a green bead on the second draw, we look at all of the outcomes that contain a green bead second. These are

- a red bead and then a green bead (RG);
- a green bead and then another green bead (GG).

The probability of the first outcome is $\frac{15}{56}$ and the probability of the second outcome is $\frac{5}{14}$. By adding these two probabilities, we see that the probability of getting a green bead second is

$$P(\text{second bead green}) = \frac{15}{56} + \frac{5}{14} = \frac{15}{56} + \frac{20}{56} = \frac{35}{56} = \frac{5}{8}$$

Step 4: Probability of red first and green second

We have already calculated the probability that the first bead is red and the second bead is green (RG). It is $\frac{15}{56}$.

Step 5: Dependent or independent?

According to the definition, events are independent if and only if

$$P(A \text{ and } B) = P(A) \times P(B)$$

In this problem:

- $P(\text{first bead red}) = \frac{3}{8}$
- $P(\text{second bead green}) = \frac{5}{8}$
- $P(\text{first bead red and second bead green}) = \frac{15}{56}$

Since $\frac{3}{8} \times \frac{5}{8} = \frac{15}{64} \neq \frac{15}{56}$, the events are dependent.

Worked example 4: The addition rule for 2 mutually exclusive events

QUESTION

A sample space, S, consists of all natural numbers less than 16. A is the event of drawing an even number at random. B is the event of randomly drawing a prime number. Are A and B mutually exclusive events? Prove this using the addition rule.

SOLUTION

Step 1: Write down the sample space

The sample space contains all the natural numbers less than 16.

$$S = \{1; 2; 3; 4; 5; 6; 7; 8; 9; 10; 11; 12; 13; 14; 15\}$$

Step 2: Write down the events

The even natural numbers less than 16 are

$$A = \{2; 4; 6; 8; 10; 12; 14\}$$

The prime numbers less than 16 are

$$B = \{2; 3; 5; 7; 11; 13\}$$

We can already see from writing down our event sets that A and B share the event 2 and are thus not mutually exclusive. However, the question asked us to prove this using the addition rule so let's go ahead and do that.

Step 3: Compute the probabilities

The probability of an event is the number of outcomes in the event set divided by the number of outcomes in the sample space. There are 15 outcomes in the sample space.

1. Since there are 7 outcomes in the A event set, $P(A) = \frac{n(A)}{n(S)} = \frac{7}{15}$.
2. Since there are 6 outcomes in the B event set, $P(B) = \frac{n(B)}{n(S)} = \frac{6}{15} = \frac{2}{5}$.
3. The event that is a prime number or an even number is the union of the above two event sets. There are 12 elements in the union of the event sets, so $P(A \text{ or } B) = \frac{n(A \text{ or } B)}{n(S)} = \frac{12}{15}$.

Step 4: Are the two events mutually exclusive?

To test whether two events are mutually exclusive, we can use the **addition rule**. For two mutually exclusive events,

$$P(A \text{ and } B) \text{ is an empty set, therefore } P(A \text{ or } B) = P(A) + P(B)$$

$$\text{Since } P(A \text{ or } B) = \frac{12}{15} \text{ and } P(A) + P(B) = \frac{6}{15} + \frac{7}{15} = \frac{13}{15}$$

$$P(A \text{ or } B) \neq P(A) + P(B)$$

Therefore the intersection of A and B is nonzero. This means that the events A and B are not mutually exclusive.

Worked example 5: The addition rule

QUESTION

The probability that a person drinks tea is $0{,}5$. The probability that a person drinks coffee is $0{,}4$. The probability that a person drinks tea, coffee or both is $0{,}8$. Determine the probability that a person drinks tea and coffee.

SOLUTION

Step 1: Determine if the events are mutually exclusive

Let the probability that a person drinks tea $= P(T)$ and the probability that a person drinks coffee $= P(C)$.

From the information provided in the question, we know that:

- $P(T) = 0{,}5$
- $P(C) = 0{,}4$
- $P(T \text{ or } C) = 0{,}8$
- $P(T) + P(C) = 0{,}5 + 0{,}4 = 0{,}9$
- Therefore $P(T \text{ or } C) \neq P(T) + P(C)$

Therefore the events are not mutually exclusive.

Step 2: Compute the probability that a person drinks tea and coffee

Using the addition rule, we know that:

$$\begin{aligned} P(A \text{ or } B) &= P(A) + P(B) - P(A \text{ and } B) \\ \therefore P(T \text{ or } C) &= P(T) + P(C) - P(T \text{ and } C) \\ 0{,}8 &= 0{,}4 + 0{,}5 - P(T \text{ and } C) \\ \therefore P(T \text{ and } C) &= 0{,}4 + 0{,}5 - 0{,}8 = 0{,}1 \end{aligned}$$

Therefore the probability that a person drinks tea and coffee is $0{,}1$.

See video: 29FT at www.everythingmaths.co.za

Worked example 6: The complementary rule

QUESTION

Joe wants to open a tuckshop at his school but is not sure which cool drinks to stock. Before opening, he interviewed a sample of learners to determine what types of cool drinks they like. From his research, he determined that the probability that a learner drinks cola is $0{,}3$, the probability that a learner drinks lemonade is $0{,}6$ and the probability that a learner drinks neither is $0{,}2$. Determine:

- the probability that a learner drinks cola and lemonade.
- the probability that a learner drinks only cola or only lemonade.

SOLUTION

Step 1: Determine the probability that a learner drinks cola or lemonade

Let the probability that a learner drinks cola = $P(C)$ and the probability that a learner drinks lemonade = $P(L)$.

From the information provided in the question, we know that:

- $P(C) = 0{,}3$
- $P(L) = 0{,}6$
- $P(\text{not } (C \text{ or } L)) = 0{,}2$

Using the complementary rule:

$$\begin{aligned} P(\text{not } (C \text{ or } L)) &= 1 - P(C \text{ or } L) \\ \therefore P(C \text{ or } L) &= 1 - P(\text{not } (C \text{ or } L)) \\ &= 1 - 0{,}2 \\ &= 0{,}8 \end{aligned}$$

Step 2: Calculate the probability that a learner drinks cola and lemonade

Using the addition rule:

$$\begin{aligned} P(C \text{ or } L) &= P(C) + P(L) - P(C \text{ and } L) \\ \therefore P(C \text{ and } L) &= P(C) + P(L) - P(C \text{ or } L) \\ &= 0{,}3 + 0{,}6 - 0{,}8 \\ &= 0{,}1 \end{aligned}$$

The probability that a learner drinks both cola and lemonade is $0{,}1$.

Step 3: Determine the probability that a learner drinks only cola or only lemonade

This question requires us to calculate the probability that a learner likes lemonade or cola but not both of them. We can write this as:

$$P(\text{only } C \text{ or only } L) = P(C \text{ or } L) - P(C \text{ and } L)$$

since a learner can like either cola or lemonade but not both.

We already know $P(C \text{ or } L) = 0{,}8$ and $P(C \text{ and } L) = 0{,}1$, therefore the probability of a learner drinking only cola or only lemonade is $0{,}7$.

Exercise 10 – 1: The product and addition rules

1. Determine whether the following events are dependent or independent and give a reason for your answer:

   a) Joan has a box of yellow, green and orange sweets. She takes out a yellow sweet and eats it. Then, she chooses another sweet and eats it.

   b) Vuzi throws a die twice.

   c) Celia chooses a card at random from a deck of 52 cards. She is unhappy with her choice, so she places the card back in the deck, shuffles it and chooses a second card.

d) Thandi has a bag of beads. She randomly chooses a yellow bead, looks at it and then puts it back in the bag. Then she randomly chooses another bead and sees that it is red and puts it back in the bag.

e) Mark has a container with calculators. Some of them work and some are broken. He randomly chooses a calculator and sees that it does not work and throws it away. He then chooses another calculator, sees that it works and keeps it.

2. Given that $P(A) = 0{,}7$; $P(B) = 0{,}4$ and $P(A \text{ and } B) = 0{,}28$,

   a) are events A and B mutually exclusive? Give a reason for your answer.

   b) are the events A and B independent? Give a reason for your answer.

3. In the following examples, are A and B dependent or independent?

   a) $P(A) = 0{,}2; P(B) = 0{,}7$ and $P(A \text{ and } B) = 0{,}21$

   b) $P(A) = 0{,}2; P(B) = 0{,}7$ and $P(B \text{ and } A) = 0{,}14$.

4. $n(A) = 5; n(B) = 4; n(S) = 20$ and $n(A \text{ or } B) = 8$.

   a) Are A and B mutually exclusive?

   b) Are A and B independent?

5. Simon rolls a die twice. What is the probability of getting:

   a) two threes.

   b) a prime number then an even number.

   c) no threes.

   d) only one three.

   e) at least one three.

6. The Mandalay Secondary soccer team has to win both of their next two matches in order to qualify for the finals. The probability that Mandalay Secondary will win their first soccer match against Ihlumelo High is $\frac{2}{5}$ and the probability of winning their second soccer match against Masiphumelele Secondary is $\frac{3}{7}$. Assume each match is an independent event.

   a) What is the probability they will progress to the finals?

   b) What is the probability they will not win either match?

   c) What is the probability they will win only one of their matches?

   d) You were asked to assume that the matches are independent events but this is unlikely in reality. What are some factors you think may result in the outcome of the matches being dependent?

7. A pencil bag contains 2 red pens and 4 green pens. A pen is drawn from the bag and then replaced before a second pen is drawn. Calculate:

   a) The probability of drawing a red pen first if a green pen is drawn second.

   b) The probability of drawing a green pen second if the first pen drawn was red.

   c) The probability of drawing a red pen first and a green pen second.

8. A lunch box contains 4 sandwiches and 2 apples. Vuyele chooses a food item randomly and eats it. He then chooses another food item randomly and eats that. Determine the following:

   a) The probability that the first item is a sandwich.

   b) The probability that the first item is a sandwich and the second item is an apple.

c) The probability that the second item is an apple.

d) Are the events in a) and c) dependent? Confirm your answer with a calculation.

9. Given that $P(A) = 0{,}5$; $P(B) = 0{,}4$ and $P(A \text{ or } B) = 0{,}7$, determine by calculation whether events A and B are:

    a) mutually exclusive

    b) independent

10. A and B are two events in a sample space where $P(A) = 0{,}3$; $P(A \text{ or } B) = 0{,}8$ and $P(B) = k$. Determine the value of k if:

    a) A and B are mutually exclusive

    b) A and B are independent

11. A and B are two events in sample space S where $n(S) = 36$; $n(A) = 9$; $n(B) = 4$ and $n(\text{not } (A \text{ or } B)) = 24$. Determine:

    a) $P(A \text{ or } B)$

    b) $P(A \text{ and } B)$

    c) whether events A and B independent. Justify your answer with a calculation.

12. The probability that a Mathematics teacher is absent from school on a certain day is $0{,}2$. The probability that the Science teacher will be absent that same day is $0{,}3$.

    a) Do you think these two events are independent? Give a reason for your answer.

    b) Assuming the events are independent, what is the probability that the Mathematics teacher or the Science teacher is absent?

    c) What is the probability that neither the Mathematics teacher nor the Science teacher is absent?

13. Langa Cricket Club plays two cricket matches against different clubs. The probability of winning the first match is $\frac{3}{5}$ and the probability of winning the second match is $\frac{4}{9}$. Assuming the results of the matches are independent, calculate the probability that Langa Cricket Club will:

    a) win both matches.

    b) not win the first match.

    c) win one or both of the two matches.

    d) win neither match.

    e) not win the first match and win the second match.

14. Two teams are working on the final problem at a Mathematics Olympiad. They have 10 minutes remaining to finish the problem. The probability that team A will finish the problem in time is 40% and the probability that team B will finish the problem in time is 25%. Calculate the probability that both teams will finish before they run out of time.

15. Thabo and Julia were arguing about whether people prefer tea or coffee. Thabo suggested that they do a survey to settle the dispute. In total, they surveyed 24 people and found that 8 of them preferred to drink coffee and 12 of them preferred to drink tea. The number of people who drink tea, coffee or both is 16. Determine:

    a) the probability that a person drinks tea, coffee or both.

b) the probability that a person drinks neither tea nor coffee.

c) the probability that a person drinks coffee and tea.

d) the probability that a person does not drink coffee.

e) whether the event that a person drinks coffee and the event that a person drinks tea are independent.

16. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 29FV	2. 29FW	3. 29FX	4. 29FY	5. 29FZ	6. 29G2
7. 29G3	8. 29G4	9. 29G5	10. 29G6	11. 29G7	12. 29G8
13. 29G9	14. 29GB	15. 29GC			

10.3 Tools and Techniques EMCJY

Venn diagrams are used to show how events are related to one another. A Venn diagram can be very helpful when doing calculations with probabilities. In a Venn diagram each event is represented by a shape, often a circle or a rectangle. The region inside the shape represents the outcomes included in the event and the region outside the shape represents the outcomes that are not in the event.

Tree diagrams are useful for organising and visualising the different possible outcomes of a sequence of events. Each branch in the tree shows an outcome of an event, along with the probability of that outcome. For each possible outcome of the first event, we draw a line where we write down the probability of that outcome and the state of the world if that outcome happened. Then, for each possible outcome of the second event we do the same thing. The probability of a sequence of outcomes is calculated as the product of the probabilities along the branches of the sequence.

Two-way contingency tables are a tool for keeping a record of the counts or percentages in a probability problem. Two-way contingency tables are especially helpful for figuring out whether events are dependent or independent.

Worked example 7: Venn diagrams

QUESTION

There are 200 boys in Grade 12 at Marist Brothers High School. Their participation in sport can be broken down as follows:

- 107 play rugby
- 90 play soccer
- 63 play cricket
- 35 play rugby and soccer
- 23 play rugby and cricket
- 15 play rugby, soccer and cricket
- 190 boys play rugby, soccer or cricket

1. How many boys do not play any of these sports?
2. Draw a Venn diagram to illustrate the given information and use it to answer the following questions:
   a) How many boys play soccer and cricket, but not rugby?
   b) What is the probability that a randomly chosen Grade 12 boy at Marist Brothers High School will take part in at least two of the sports: rugby, soccer or cricket? Give your answer correct to 3 decimal places.

SOLUTION

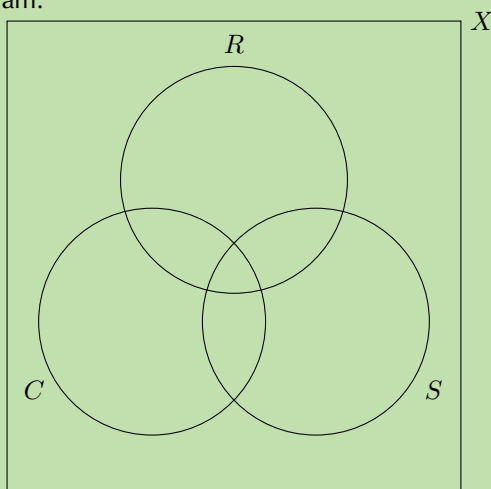
Step 1: Calculate the number of boys playing none of the given sports

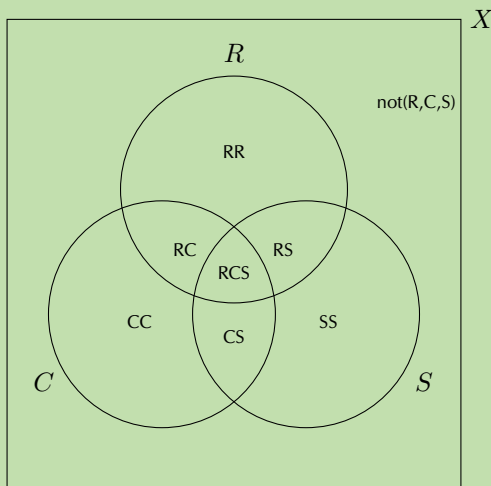
In order to calculate the number of boys playing none of the sports, we subtract the number of boys playing any of the three sports from the total number of boys in the sample space.

$$\text{Not rugby, cricket or soccer } = 200 - 190 = 10$$

Therefore 10 boys do not play rugby, cricket or soccer.

Step 2: Draw the outline of the Venn diagram

Let X = the sample space; R = rugby; S = soccer and C = cricket. Put this information on a Venn diagram:

Step 3: Calculate the counts for the different groupings

The following groupings exist:

Rugby, cricket and soccer: RCS

$$RCS = 15$$

Rugby and soccer but not cricket: RS

$$\begin{aligned} RS &= (R \text{ and } S) - RCS \\ &= 35 - 15 = 20 \end{aligned}$$

Rugby and cricket but not soccer: RC

$$\begin{aligned} RC &= (R \text{ and } C) - RCS \\ &= 23 - 15 = 8 \end{aligned}$$

Cricket and soccer but not rugby: CS

$$CS = (S \text{ and } C) - RCS$$

Let $(S \text{ and } C) = x$

Therefore $CS = x - 15$

Only rugby: RR

$$\begin{aligned} RR &= R - RS - RC - RCS \\ &= 107 - 20 - 8 - 15 \\ &= 64 \end{aligned}$$

Only soccer: SS

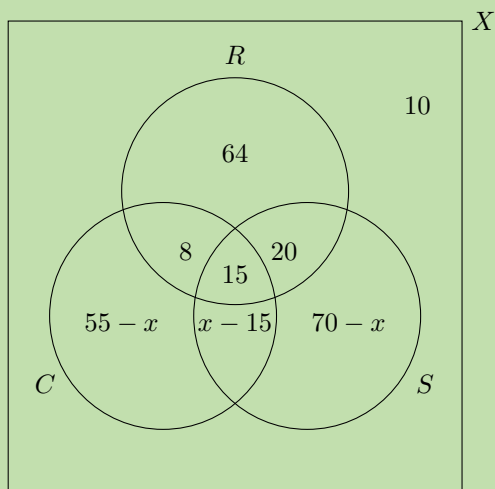
$$\begin{aligned} SS &= S - RS - CS - RCS \\ &= 90 - 20 - (x - 15) - 15 \\ &= 70 - x \end{aligned}$$

Only cricket: CC

$$\begin{aligned} CC &= C - RC - CS - RCS \\ &= 63 - 8 - (x - 15) - 15 \\ &= 55 - x \end{aligned}$$

Not rugby, cricket or soccer: not(R, C, S)

$$\text{not}(R, C, S) = 10$$

Step 4: Fill in the counts on the Venn diagram

Step 5: Calculate the unknown values

Since 190 of the boys play at least one of the sports, using the values on our Venn diagram, we can set up the following equation to solve for x.

$$\begin{aligned} 64 + 8 + 15 + 20 + (x - 15) + (70 - x) + (55 - x) &= 190 \\ 217 - x &= 190 \\ \text{Therefore } x &= 27 \end{aligned}$$

We know that:

$$\begin{aligned} \text{Cricket and soccer but not rugby } (CS) &= x - 15 \\ \text{Therefore } CS &= 27 - 15 \\ &= 12 \end{aligned}$$

Therefore there are 12 boys who play cricket and soccer but not rugby.

Step 6: Calculate the probability that a randomly chosen Grade 12 boy plays at least two of the given sports

We know the number of boys who play two or more of rugby, cricket or soccer and we know the total number of boys. Therefore, we can calculate the probability using the following equation:

$$\begin{aligned} P(\text{at least two sports}) &= \frac{n(RC) + n(RS) + n(CS) + n(RCS)}{n(X)} \\ &= \frac{8 + 15 + 20 + 12}{200} \\ &= \frac{55}{200} = 0{,}275 \end{aligned}$$

Therefore the probability that a randomly chosen Grade 12 boy plays at least 2 of either rugby, cricket or soccer $= 0{,}275$ or $27{,}5\%$

See video: 29GD at www.everythingmaths.co.za

Worked example 8: Tree diagrams

QUESTION

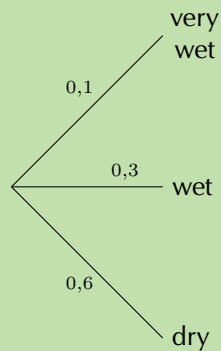
The probability that the floor of a supermarket will be wet when it opens in the morning is 30% and there is a 10% probability of the floor being very wet. The probability that a person will slip and fall if the floor is dry is 12% and a person is three times as likely to fall if the floor is wet. If the floor is very wet, the probability that a person will fall is $0{,}6$. Draw a tree diagram to represent the given information, showing the probabilities of each outcome, and use it to answer the following questions:

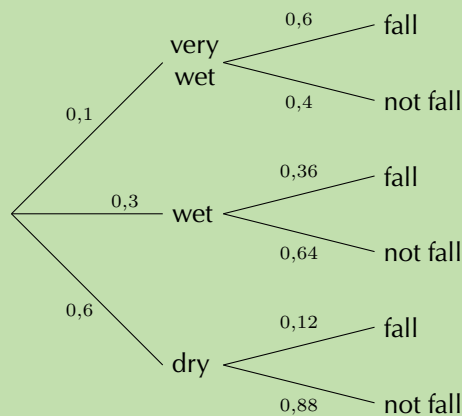
1. What is the probability that a person will fall on any given day?
2. What is the probability that a person will not fall on any given day?
3. Are the events of the floor being dry and a person falling independent? Justify your answer with a calculation.

SOLUTION

Step 1: Identify the events

There are three outcomes for the floor, namely, dry, wet and very wet, and two outcomes for a person, namely fall or not fall.

Step 2: Draw the first level of the tree diagram

This tree diagram shows the possible outcomes and probabilities of the status of the floor.

Step 3: Draw the second level of the tree diagram

This tree diagram shows the possible outcomes and probabilities based on whether the floor is very wet, wet or dry. Remember that the sum of the probabilities for any set of branches is 1. Use this as a logical check whenever you are constructing a tree diagram.

Step 4: Compute the probabilities of the various outcomes

We can calculate the probability of each outcome by multiplying the probabilities along the path from the start of the tree to the end of the branch containing the desired outcome.

- $P(\text{very wet and fall}) = 0{,}1 \times 0{,}6 = 0{,}06$
- $P(\text{very wet and not fall}) = 0{,}1 \times 0{,}4 = 0{,}04$
- $P(\text{wet and fall}) = 0{,}3 \times 0{,}36 = 0{,}108$
- $P(\text{wet and not fall}) = 0{,}3 \times 0{,}64 = 0{,}192$
- $P(\text{dry and fall}) = 0{,}6 \times 0{,}12 = 0{,}072$
- $P(\text{dry and not fall}) = 0{,}6 \times 0{,}88 = 0{,}528$

Step 5: Compute the probability of falling or not falling

We can calculate the probability of falling or not falling by adding the probabilities of all the desired outcomes.

- $P(\text{fall}) = 0{,}06 + 0{,}108 + 0{,}072 = 0{,}24$
- $P(\text{not fall}) = 0{,}04 + 0{,}192 + 0{,}528 = 0{,}76$

Therefore the probability of falling on a given day is 24% and the probability of not falling is 76%.

Step 6: Determine whether the floor being dry and a person falling are independent events

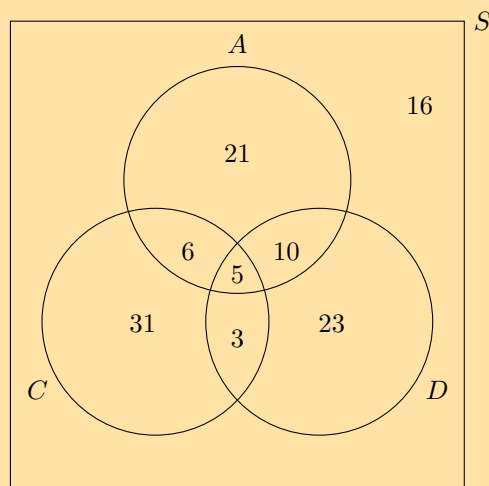
Logically, it appears that these events are dependent but the question asked us to prove this using a calculation. We can do this using the rule for independent events:

$$P(A \text{ and } B) = P(A) \times P(B)$$

$$\begin{aligned} P(\text{dry and fall}) &= 0{,}072 \\ P(\text{dry}) \times P(\text{fall}) &= 0{,}6 \times 0{,}24 \\ &= 0{,}144 \\ \text{Therefore } P(\text{dry and fall}) &\neq P(\text{dry}) \times P(\text{fall}) \end{aligned}$$

Therefore we can conclude that the floor being dry and a person falling are dependent events.

Exercise 10 – 2: Venn and tree diagrams

1. A survey was done on a group of learners to determine which type of TV shows they enjoy: action, comedy or drama. Let A = action, C = comedy and D = drama. The results of the survey are shown in the Venn diagram below.

Study the Venn diagram and determine the following:

a) the total number of learners surveyed
b) the number of learners who do not enjoy any of the mentioned types of TV shows
c) $P(\text{not } A)$
d) $P(A \text{ or } D)$
e) $P(A \text{ and } C \text{ and } D)$
f) $P(\text{not } (A \text{ and } D))$
g) $P(A \text{ or not } C)$
h) $P(\text{not } (A \text{ or } C))$
i) the probability of a learner enjoying at least two types of TV shows
j) Describe, in words, the meaning of each of the questions c) to h) in the context of this problem.

2. At Thandokulu Secondary School, there are are 320 learners in Grade 12, 270 of whom take one or more of Mathematics, History and Economics. The subject choice is such that everybody who takes Physical Sciences must also take Mathematics and nobody who takes Physical Sciences can take History or Economics. The following is known about the number of learners who take these subjects:

- 70 take History
- 50 take Economics
- 120 take Physical Sciences
- 200 take Mathematics
- 20 take Mathematics and History
- 10 take History and Economics
- 25 take Mathematics and Economics
- x learners take Mathematics and History and Economics

a) Represent the information above in a Venn diagram. Let Mathematics be M, History be H, Physical Sciences be P and Economics be E.
b) Determine the number of learners, x, who take Mathematics, History and Economics.
c) Determine $P(\text{not } (M \text{ or } H \text{ or } E))$ and state in words what your answer means.
d) Determine the probability that a learner takes at least two of these subjects.

3. A group of 200 people were asked about the kind of sports they watch on television. The information collected is given below:

- 180 watch rugby, cricket or soccer
- 5 watch rugby, cricket and soccer
- 25 watch rugby and cricket
- 30 watch rugby and soccer
- 100 watch rugby
- 65 watch cricket
- 80 watch soccer
- x watch cricket and soccer but not rugby

a) Represent all the information in a Venn diagram. Let rugby watchers $= R$, cricket watchers $= C$ and soccer watchers $= F$.
b) Find the value of x.
c) Determine P(not $(R$ or F or $C))$
d) Determine $P(R$ or F or not $C)$
e) Are watching cricket and watching rugby independent events? Confirm your answer using a calculation.

4. There are 25 boys and 15 girls in the English class. Each lesson, two learners are randomly chosen to do an oral.

   a) Represent the composition of the English class in a tree diagram. Include all possible outcomes and probabilities.
   b) Calculate the probability that a boy and a girl are chosen to do an oral in any particular lesson.
   c) Calculate the probability that at least one of the learners chosen to do an oral in any particular lesson is male.
   d) Are the events picking a boy first and picking a girl second independent or dependent? Justify your answer with a calculation.

5. During July in Cape Town, the probability that it will rain on a randomly chosen day is $\frac{4}{5}$. Gladys either walks to school or gets a ride with her parents in their car. If it rains, the probability that Gladys' parents will take her to school by car is $\frac{5}{6}$. If it does not rain, the probability that Gladys' parents will take her to school by car is $\frac{1}{12}$.

   a) Represent the above information in a tree diagram. On your diagram show all the possible outcomes and respective probabilities.
   b) What is the probability that it is a rainy day and Gladys walks to school?
   c) What is the probability that Gladys' parents take her to school by car?

6. There are two types of property burglaries: burglary of private residences and burglary of business premises. In Metropolis, burglary of a private residence is four times as likely as that of a business premises. The following statistics for each type of burglary were obtained from the Metropolis Police Department:

   Burglary of private residences

   Following a burglary:

   - 25% of criminals are arrested within 48 hours.
   - 15% of criminals are arrested after 48 hours.
   - 60% of criminals are never arrested for that particular burglary.

   Burglary of business premises

   Following a burglary:

   - 36% of criminals are arrested within 48 hours.
   - 54% of criminals are arrested after 48 hours.
   - 10% of criminals are never arrested for that particular burglary.

   a) Represent the information above in a tree diagram, showing all outcomes and respective probabilities.
   b) Calculate the probability that a private home is burgled and nobody is arrested.
   c) Calculate the probability that burglars of private homes and business premises are arrested.

d) Use your answer in the previous question to construct a tree diagram to calculate the probability that a burglar is arrested after at most three burglaries.

e) Determine after how many burglaries a burglar has at least a

i. 90% chance of being arrested.

ii. 99% chance of being arrested.

7. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 29GF 2. 29GG 3. 29GH 4. 29GJ 5. 29GK 6. 29GM

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Worked example 9: Two-way contingency tables

QUESTION

The table below shows the results of testing two different treatments on 240 fruit trees which have a disease causing the trees to die. Treatment A involves the careful removal of infected branches and treatment B involves removing infected branches as well as spraying the tree with antibiotic.

	Tree dies within 4 years	Tree lives > 4 years	Total
Treatment A	70	50	
Treatment B			
Total	90	150	

1. Fill in the missing values on the table.
2. What is the probability the a tree received treatment B?
3. What is the probability that a tree will live beyond 4 years?
4. What is the probability that a tree is given treatment B and lives beyond 4 years?
5. Of the trees who were given treatment B, what is the probability that a tree lives beyond 4 years?
6. Are a tree given treatment B and living beyond 4 years independent events? Justify your answer with a calculation.

SOLUTION

Step 1: Complete the contingency table

Since each column has to sum to its total, we can work out the number of trees which fall into each category for treatments A and B. Then, we can add each row to get the totals on the right hand side of the table.

	Tree dies within 4 years	Tree lives > 4 years	Total
Treatment A	70	50	120
Treatment B	20	100	120
Total	90	150	240

Step 2: Compute the required probabilities

For the second question, we need to determine the probability that a tree receives treatment B. This means that we do not include treatment A in this calculation. So, the probability that treatment B is given to a tree is the ratio between the number of trees that received treatment B and the total number of trees.

$$\begin{aligned} P(\text{treatment B}) &= \frac{n(\text{treatment B})}{n(\text{total trees})} \\ &= \frac{120}{240} = \frac{1}{2} \end{aligned}$$

Similarly for the third question, the probability that a tree will live beyond 4 years:

$$\begin{aligned} P(\text{lives beyond 4 years}) &= \frac{n(\text{lives } > 4 \text{ years})}{n(\text{total trees})} \\ &= \frac{150}{240} = \frac{5}{8} \end{aligned}$$

In the fourth question, we need to determine the probability that a tree receives treatment B and lives beyond 4 years.

$$\begin{aligned} P(\text{treatment B and lives } > 4 \text{ years}) &= \frac{n(\text{treatment B and lives } > 4 \text{ years})}{n(\text{total trees})} \\ &= \frac{100}{240} = \frac{5}{12} \end{aligned}$$

In the fifth question, there is a subtle change from the fourth question. Here, we need to determine the probability that of the trees which received treatment B, a tree lives beyond 4 years. This means we are only concerned with those trees which received treatment B. We no longer need to care about the trees given treatment A, so our denominator needs to be adjusted accordingly.

$$\begin{aligned} P(\text{lives } > 4 \text{ years having received treatment B}) &= \frac{n(\text{treatment B and lives } > 4 \text{ years})}{n(\text{total treatment B})} \\ &= \frac{100}{120} = \frac{5}{6} \end{aligned}$$

Step 3: Independence

We need to determine whether a tree given treatment B and living beyond 4 years are dependent or independent events. According to the definition, two events are independent if and only if

$$P(A \text{ and } B) = P(A) \times P(B)$$

$$P(\text{treatment B}) \times P(\text{lives } > 4 \text{ years}) = \frac{1}{2} \times \frac{5}{8} = \frac{5}{16}$$

$$P(\text{treatment B and lives } > 4 \text{ years}) = \frac{5}{12}$$

From these probabilities we can see that

$$P(\text{treatment B and lives } > 4 \text{ years}) \neq P(\text{treatment B}) \times P(\text{lives } > 4 \text{ years})$$

and therefore the treatment of a tree with treatment B and living beyond 4 years are dependent events.

Exercise 10 – 3: Contingency tables

1. A number of drivers were asked about the number of motor vehicle accidents they were involved in over the last 10 years. Part of the data collected is shown in the table below.

	$\leq$ 2 accidents	$>$ 2 accidents	Total
Female	210	90	
Male			
Total	350	150	500

   a) What are the variables investigated here and what is the purpose of the research?

   b) Complete the table.

   c) Determine whether gender and number of accidents are independent using a calculation.

2. Researchers conducted a study to test how effective a certain inoculation is at preventing malaria. Part of their data is shown below:

	Malaria	No malaria	Total
Male	a	b	216
Female	c	d	648
Total	108	756	864

   a) Calculate the probability that a randomly selected study participant will be female.

   b) Calculate the probability that a randomly selected study participant will have malaria.

   c) If being female and having malaria are independent events, calculate the value c.

   d) Using the value of c, fill in the missing values on the table.

3. The reaction time of 400 drivers during an emergency stop was tested. Within the study cohort (the group of people being studied), the probability that a driver chosen at random was 40 years old or younger is 0,3 and the probability of a reaction time less than 1,5 seconds is 0,7.

   a) Calculate the number of drivers who are 40 years old or younger.

   b) Calculate the number of drivers who have a reaction time of less than 1,5 seconds.

   c) If age and reaction time are independent events, calculate the number of drivers 40 years old and younger with a reaction time of less than 1,5 seconds.

   d) Complete the table below.

	Reaction time $<$ 1,5 s	Reaction time $>$ 1,5 s	Total
$\leq$40 years			
$>$ 40 years			
Total			400

4. A new treatment for influenza (the flu) was tested on a number of patients to determine if it was better than a placebo (a pill with no therapeutic value). The table below shows the results three days after treatment:

	Flu	No flu	Total
Placebo	228	60	
Treatment			
Total	240	312	

   a) Complete the table.
   b) Calculate the probability of a patient receiving the treatment.
   c) Calculate the probability of a patient having no flu after three days.
   d) Calculate the probability of a patient receiving the treatment and having no flu after three days.
   e) Using a calculation, determine whether a patient receiving the treatment and having no flu after three days are dependent or independent events.
   f) Calculate the probability that a patient receiving treatment will have no flu after three days.
   g) Calculate the probability that a patient receiving a placebo will have no flu after three days.
   h) Comparing you answers in f) and g), would you recommend the use of the new treatment for patients suffering from influenza?
   i) A hospital is trying to decide whether to purchase the new treatment. The new treatment is much more expensive than the old treatment. According to the hospital records, of the $72\ 024$ flu patients that have been treated with the old treatment, only 3200 still had the flu three days after treatment.
      - Construct a two-way contingency table comparing the old treatment data with the new treatment data.
      - Using the data from your table, advise the hospital whether to purchase the new treatment or not.

5. Human immunodeficiency virus (HIV) affects 10% of the South African population.
   a) If a test for HIV has a $99{,}9\%$ accuracy rate (i.e. $99{,}9\%$ of the time the test is correct, $0{,}1\%$ of the time, the test returns a false result), draw a two-way contingency table showing the expected results if $10\ 000$ of the general population are tested.
   b) Calculate the probability that a person who tests positive for HIV does not have the disease, correct to two decimal places.
   c) In practice, a person who tests positive for HIV is always tested a second time. Calculate the probability that an HIV-negative person will test positive after two tests, correct to four decimal places.

6. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 29GN 2. 29GP 3. 29GQ 4. 29GR 5. 29GS

10.4 The fundamental counting principle EMCJZ

Mathematics began with counting. Initially, fingers, beans and buttons were used to help with counting, but these are only practical for small numbers. What happens when a large number of items must be counted?

This section focuses on how to use mathematical techniques to count different assortments of items.

Introduction EMCK2

An important aspect of probability theory is the ability to determine the total number of possible outcomes when multiple events are considered.

For example, what is the total number of possible outcomes when a die is rolled and then a coin is tossed? The roll of a die has six possible outcomes ($1; 2; 3; 4; 5$ or 6) and the toss of a coin, 2 outcomes (heads or tails). The sample space (total possible outcomes) can be represented as follows:

$$S = \left\{ \begin{array}{cccccc} (1;H); & (2;H); & (3;H); & (4;H); & (5;H); & (6;H); \\ (1;T); & (2;T); & (3;T); & (4;T); & (5;T); & (6;T) \end{array} \right\}$$

Therefore there are 12 possible outcomes.

The use of lists, tables and tree diagrams is only feasible for events with a few outcomes. When the number of outcomes grows, it is not practical to list the different possibilities and the fundamental counting principle is used instead.

DEFINITION: *The fundamental counting principle*

The fundamental counting principle states that if there are $n(A)$ outcomes in event A and $n(B)$ outcomes in event B, then there are $n(A) \times n(B)$ outcomes in event A and event B combined.

If we apply this principle to our previous example, we can easily calculate the number of possible outcomes by multiplying the number of possible die rolls with the number of outcomes of tossing a coin: $6 \times 2 = 12$ outcomes. This allows us to formulate the following:

If there n_1 possible outcomes for event A and n_2 outcomes for event B, then the total possible number of outcomes for both events is $n_1 \times n_2$

This can be generalised to k events, where k is the number of events. The total number of outcomes for k events is:

$$n_1 \times n_2 \times n_3 \times \cdots \times n_k$$

NOTE:
The order in which the experiments are done does not affect the total number of possible outcomes.

Worked example 10: Choices without repetition

QUESTION

A take-away has a 4-piece lunch special which consists of a sandwich, soup, dessert and drink for R $25{,}00$. They offer the following choices for:

Sandwich: chicken mayonnaise, cheese and tomato, tuna mayonnaise, ham and lettuce

Soup: tomato, chicken noodle, vegetable

Dessert: ice-cream, piece of cake

Drink: tea, coffee, Coke, Fanta, Sprite

How many possible meals are there?

SOLUTION

Step 1: Determine how many parts to the meal there are
There are 4 parts: sandwich, soup, dessert and drink.

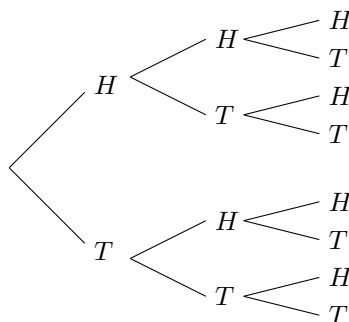
Step 2: Identify how many choices there are for each part

Meal component	Sandwich	Soup	Dessert	Drink
Number of choices	4	3	2	5

Step 3: Use the fundamental counting principle to determine how many different meals are possible
$4 \times 3 \times 2 \times 5 = 120$

So there are 120 possible meals.

In the previous example, there were a different number of options for each choice. But what happens when the number of choices is unchanged each time you choose?

For example, if a coin is flipped three times, what is the total number of different results? Each time a coin is flipped, there are two possible outcomes, namely heads or tails. The coin is flipped 3 times. We can use a tree diagram to determine the total number of possible outcomes:

From the tree diagram, we can see that there is a total of 8 different possible outcomes.

Drawing a tree diagram is possible for three different coin flips, but as soon as the number of events increases, the total number of possible outcomes increases to the point where drawing a tree diagram is impractical.

For example, think about what a tree diagram would look like if we were to flip a coin six times. In this case, using the fundamental counting principle is a far easier option. We know that each time a coin is flipped that there are two possible outcomes. So if we flip a coin six times, the total number of possible outcomes is equivalent to multiplying 2 by itself six times:

$$2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^6 = 64$$

Another example is if you have the letters A, B, C, and D and you wish to discover the number of ways of arranging them in three-letter patterns if repetition is allowed, such as ABA, DCA, BBB etc. You will find that there are 64 ways. This is because for the first letter of the pattern, you can choose any of the four available letters, for the second letter of the pattern, you can choose any of the four letters, and for the final letter of the pattern you can choose any of the four letters. Multiplying the number of available choices for each letter in the pattern gives the total available arrangements of letters:

$$4 \times 4 \times 4 = 4^3 = 64$$

This allows us to formulate the following:

When you have n objects to choose from and you choose from them r times, then the total number of possibilities is

$$n \times n \times n \ldots \times n \ \ (r \text{ times}) = n^r$$

Worked example 11: Choices with repetition

QUESTION

A school plays a series of 6 soccer matches. For each match there are 3 possibilities: a win, a draw or a loss. How many possible results are there for the series?

SOLUTION

Step 1: Determine how many outcomes you have to choose from for each event

There are 3 outcomes for each match: win, draw or lose.

Step 2: Determine the number of events

There are 6 matches, therefore the number of events is 6.

Step 3: Determine the total number of possible outcomes

There are 3 possible outcomes for each of the 6 events. Therefore, the total number of possible outcomes for the series of matches is

$$3 \times 3 \times 3 \times 3 \times 3 \times 3 = 3^6 = 729$$

Exercise 10 – 4: Number of possible outcomes if repetition is allowed

1. Tarryn has five different skirts, four different tops and three pairs of shoes. Assuming that all the colours complement each other, how many different outfits can she put together?

2. In a multiple-choice question paper of 20 questions the answers can be A, B, C or D. How many different ways are there of answering the question paper?

3. A debit card requires a five digit personal identification number (PIN) consisting of digits from 0 to 9. The digits may be repeated. How many possible PINs are there?

4. The province of Gauteng ran out of unique number plates in 2010. Prior to 2010, the number plates were formulated using the style LLLDDDGP, where L is any letter of the alphabet excluding vowels and Q, and D is a digit between 0 and 9. The new style the Gauteng government introduced is LLDDLLGP. How many more possible number plates are there using the new style when compared to the old style?

5. A gift basket consists of one CD, one book, one box of sweets, one packet of nuts and one bottle of fruit juice. The person who makes the gift basket can choose from five different CDs, eight different books, three different boxes of sweets, four kinds of nuts and six flavours of fruit juice. How many different gift baskets can be produced?

6. The code for a safe is of the form XXXXYYY where X is any number from 0 to 9 and Y represents the letters of the alphabet. How many codes are possible for each of the following cases:
   a) the digits and letters of the alphabet can be repeated.
   b) the digits and letters of the alphabet can be repeated, but the code may not contain a zero or any of the vowels in the alphabet.
   c) the digits and letters of the alphabet can be repeated, but the digits may only be prime numbers and the letters X, Y and Z are excluded from the code.

7. A restaurant offers four choices of starter, eight choices for the main meal and six choices for dessert. A customer can choose to eat just one course, two different courses or all three courses. Assuming that all courses are available, how many different meal options does the restaurant offer?

8. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 29GV 2. 29GW 3. 29GX 4. 29GY 5. 29GZ 6. 29H2
7. 29H3

10.5 Factorial notation EMCK3

Worked example 12: The arrangement of outcomes without repetition

QUESTION

Eight athletes take part in a 400 m race. In how many different ways can all 8 places in the race be arranged?

SOLUTION

Any of the 8 athletes can come first in the race. Now there are only 7 athletes left to be second, because an athlete cannot be both second and first in the race. After second place, there are only 6 athletes left for the third place, 5 athletes for the fourth place, 4 athletes for the fifth place, 3 athletes for the sixth place, 2 athletes for the seventh place and 1 athlete for the eighth place. Therefore the number of ways that the athletes can be ordered is as follows:

$$8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40\ 320$$

As in the example above, it is a common occurrence in counting problems that the outcome of the first event reduces the number of possible outcomes for the second event by exactly 1, and the outcome of the second event reduces the possible outcomes for the third event by 1 more, etc.

As this sort of problem occurs so frequently, we have a special notation to represent the answer. For an integer, n, the notation $n!$ (read n factorial) represents: $n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$

This allows us to formulate the following:

The total number of possible arrangements of n different objects is

$$n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1 = n!$$

with the following definition: 0! = 1.

Worked example 13: Factorial notation

QUESTION

1. Determine $12!$
2. Show that $\frac{8!}{4!} = 8 \times 7 \times 6 \times 5$
3. Show that $\frac{n!}{(n-1)!} = n$

SOLUTION

1. We know from the definition of a factorial that $12! = 12 \times 11 \times 10 \times \ldots \times 3 \times 2 \times 1$. However, it can be quite tedious to work this out by calculating each multiplication step on paper or typing each step into your calculator. Fortunately, there is a button on your calculator which makes this much easier. To use your calculator to work out the factorial of a number:
   - Input the number.
   - Press SHIFT on your CASIO or 2ndF on your SHARP calculator.
   - Then press $x!$ on your CASIO or $n!$ on your SHARP calculator.
   - Finally, press equals to calculate the answer.

   If we follow these steps for $12!$, we get the answer $479\ 001\ 600$.
2. Expand the factorial notation:
$$\frac{8!}{4!} = \frac{8 \times 7 \times 6 \times 5 \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}{\cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}} = 8 \times 7 \times 6 \times 5 = \text{RHS}$$
3. Expand the factorial notation:
$$\frac{n!}{(n-1)!} = \frac{n \times \cancel{(n-1)} \times \cancel{(n-2)} \times \cancel{(n-3)} \times \ldots \times \cancel{3} \times \cancel{2} \times \cancel{1}}{\cancel{(n-1)} \times \cancel{(n-2)} \times \cancel{(n-3)} \times \ldots \times \cancel{3} \times \cancel{2} \times \cancel{1}} = n$$
   If $n = 1$, we get $\frac{1!}{0!}$. This is a special case. Both $1!$ and $0! = 1$, therefore $\frac{1!}{0!} = 1$ so our identity still holds.

Exercise 10 – 5: Factorial notation

1. Work out the following without using a calculator:

   a) $3!$

   b) $6!$

   c) $2!3!$

   d) $8!$

   e) $\frac{6!}{3!}$

   f) $6! + 4! - 3!$

   g) $\frac{6! - 2!}{2!}$

   h) $\frac{2! + 3!}{5!}$

   i) $\frac{2! + 3! - 5!}{3! - 2!}$

   j) $(3!)^3$

   k) $\frac{3! \times 4!}{2!}$

2. Calculate the following using a calculator:

   a) $\frac{12!}{2!}$

   b) $\frac{10!}{20!}$

   c) $\frac{10! + 12!}{5! + 6!}$

   d) $5!(2! + 3!)$

   e) $(4!)^2(3!)^2$

3. Show that the following is true:

   a) $\dfrac{n!}{(n-2)!} = n^2 - n$

   b) $\dfrac{(n-1)!}{n!} = \dfrac{1}{n}$

   c) $\dfrac{(n-2)!}{(n-1)!} = \dfrac{1}{n-1}$ for $n > 1$

4. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 29H4 1b. 29H5 1c. 29H6 1d. 29H7 1e. 29H8 1f. 29H9
1g. 29HB 1h. 29HC 1i. 29HD 1j. 29HF 1k. 29HG 2a. 29HH
2b. 29HJ 2c. 29HK 2d. 29HM 2e. 29HN 3a. 29HP 3b. 29HQ
3c. 29HR

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10.6 Application to counting problems EMCK4

Worked example 14: Further arrangement of outcomes without repetition

QUESTION

Eight athletes take part in a 400 m race. In how many different ways can the first three places be arranged?

SOLUTION

Eight different athletes can occupy the first 3 places. For the first place, there are 8 different choices. For the second place there are 7 different choices and for the third place there are 6 different choices. Therefore 8 different athletes can occupy the first three places in:

$$8 \times 7 \times 6 = 336 \text{ ways}$$

Worked example 15: Arrangement of objects with constraints

QUESTION

In how many ways can seven boys of different ages be seated on a bench if:

1. the youngest boy sits next to the oldest boy?
2. the youngest and the oldest boys must not sit next to each other?

SOLUTION

1. This question is a little different to the previous problems of arrangements without repetition. In this question, we have the constraint that the youngest boy and the oldest boy must sit together. The easiest way to think about this, is to see each set of objects which have to be together as a single object to arrange.

   If we let boy $= B$ and let the number subscript indicate order of age, we can view the objects to arrange as follows:

   $$\begin{array}{cccccc} (B_1; B_7); & (B_2); & (B_3); & (B_4); & (B_5); & (B_6) \\ 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

   If the the youngest and oldest boys are treated as a single object, there are six different objects to arrange so there are $6!$ different arrangements. However, the youngest and oldest boys can be arranged in $2!$ different ways and still be together:

   $$(B_1; B_7) \quad \text{or} \quad (B_7; B_1)$$

   Therefore there are:

   $$6! \times 2! = 1440 \text{ ways for the boys to be seated}$$

2. The arrangements where the youngest and oldest must not sit together is the total number of arrangements minus the number of arrangements where the oldest and youngest sit together. Therefore, there are:

   $$7! - 1440 = 3600 \text{ ways for the boys to be seated}$$

Exercise 10 – 6: Number of choices in a row

1. How many different possible outcomes are there for a swimming event with six competitors?
2. How many different possible outcomes are there for the gold (1st), silver (2nd) and bronze (3rd) medals in a swimming event with six competitors?
3. Susan wants to visit her friends in Pretoria, Johannesburg, Phalaborwa, East London and Port Elizabeth. In how many different ways can the visits be arranged?
4. A head boy, a deputy head boy, a head girl and a deputy head girl must be chosen out of a student council consisting of 18 girls and 18 boys. In how many ways can they be chosen?
5. Twenty different people enter a golf competition. Only the first six of them can win prizes. In how many different ways can the prizes be won?
6. Three letters of the word 'EMPTY' are arranged in a row. How many different arrangements are possible?
7. Pool balls are numbered from 1 to 15. You have only one set of pool balls. In how many different ways can you arrange:
   a) all 15 balls. Write your answer in scientific notation, rounding off to two decimal places.
   b) four of the 15 balls.

8. The captains of all the sports teams in a school have to stand next to each other for a photograph. The school sports programme offers rugby, cricket, hockey, soccer, netball and tennis.

   a) In how many different orders can they stand in the photograph?
   b) In how many different orders can they stand in the photograph if the rugby captain stands on the extreme left and the cricket captain stands on the extreme right?
   c) In how many different orders can they stand if the rugby captain, netball captain and cricket captain must stand next to each other?

9. How many three-digit numbers can be made from the digits 1 to 6 if:

   a) repetition is not allowed?
   b) repetition is allowed?

10. There are two different red books and three different blue books on a shelf.

    a) In how many different ways can these books be arranged?
    b) If you want the red books to be together, in how many different ways can the books be arranged?
    c) If you want all the red books to be together and all the blue books to be together, in how many different ways can the books be arranged?

11. There are two different Mathematics books, three different Natural Sciences books, two different Life Sciences books and four different Accounting books on a shelf. In how many different ways can they be arranged if:

    a) the order does not matter?
    b) all the books of the same subject stand together?
    c) the two Mathematics books stand first?
    d) the Accounting books stand next to each other?

12. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 29HS 2. 29HT 3. 29HV 4. 29HW 5. 29HX 6. 29HY
7. 29HZ 8. 29J2 9. 29J3 10. 29J4 11. 29J5

Worked example 16: Arrangement of letters

QUESTION

If you take the word, 'OMO', how many letter arrangements can we make if:

1. we consider the two O's as different letters?
2. we consider the two O's as identical letters?

SOLUTION

1. Since we consider the two O's as different letters, write the first O as O_1 and the second O_2. The different letter arrangements are as follows:

$$\begin{array}{ccc} O_1MO_2 & MO_1O_2 & O_1O_2M \\ O_2MO_1 & MO_2O_1 & O_2O_1M \end{array}$$

   We can see from writing out all the arrangements that there are 6 different ways for the letters to be arranged. This is not practical if there are a large number of letters. Instead, we can work this out more easily using the fundamental counting principle.

   Using the fundamental counting principle, as there are 3 letters in the word OMO if we treat each O as a separate letter, there are $3! = 6$ different arrangements.

2. If we consider the two O's as identical letters, only 3 arrangements are possible:

   OMO MOO OOM

   We can also work this out using a modified version of the previous identity. We know that if we treat each letter as different, the number of arrangements is $3!$. However, when we have duplicate letters, we have to remove the identical arrangements of these letter from our final answer. So we divide by the factorial of the number of times a letter is repeated.

   In this example, O appears twice so we divide $3!$ by $2!$:

$$\text{number of arrangements} = \frac{3!}{2!} = 3$$

Worked example 17: The number of letter arrangements for a longer word

QUESTION

If you take the word 'BASSOON', how many letter arrangements can you make if:

1. repeated letters are treated as different?
2. repeated letters are treated as identical?
3. the word starts with an O and repeated letters are treated as identical?
4. the word starts and ends with the same letter and repeated letters are treated as identical?

SOLUTION

1. There are 7 letters in the word 'BASSOON'. If we treat each letter as a different letter, there are $7! = 5040$ arrangements.
2. If repeated letters are treated as identical characters, there are two S's and two O's. This is similar to the previous worked example except now we have more than one letter repeated. When more than one letter is repeated, we have to divide the total number of possible arrangements by the product of the factorials of the number of times each letter is repeated.

$$\text{Number of arrangements} = \frac{7!}{2! \times 2!} = 1260 \text{ arrangements}$$

3. If the word starts with an 'O', there are still 6 letters left of which two are S's.

$$\text{Number of arrangements } = \frac{6!}{2!} = 360 \text{ arrangements}$$

4. If the word starts and ends with the same letter, there are two possibilities:
   - S - - - - - S with the letters in between consisting of 'B','A','O','O' and 'N'. Therefore:

   $$\text{Number of arrangements } = \frac{5!}{2!} = 60 \text{ arrangements}$$

   - O - - - - - O with the letters in between consisting of 'B','A','S','S' and 'N'. Therefore:

   $$\text{Number of arrangements } = \frac{5!}{2!} = 60 \text{ arrangements}$$

   Therefore, the total number of arrangements $= 60 + 60 = 120$.

This allows us to formulate the following:

For a set of n objects, of which n_1 are the same, n_2 are the same $\ldots, n_k$ are the same, the number of arrangements $= \dfrac{n!}{n_1! \times n_2! \ldots n_k!}$

Exercise 10 – 7: Number of arrangements of sets containing alike objects

1. You have the word 'EXCELLENT'.
   a) If the repeated letters are regarded as different letters, how many letter arrangements are possible?
   b) If the repeated letters are regarded as identical, how many letter arrangements are possible?
   c) If the first and last letters are identical, how many letter arrangements are there?
   d) How many letter arrangements can be made if the arrangement starts with an L?
   e) How many letter arrangements are possible if the word ends in a T?
2. You have the word 'ASSESSMENT'.
   a) If the repeated letters are regarded as different letters, how many letter arrangements are possible?
   b) If the repeated letters are regarded as identical, how many letter arrangements are possible?
   c) If the first and last letters are identical, how many letter arrangements are there?
   d) How many letter arrangements can be made if the arrangement starts with a vowel?
   e) How many letter arrangements are possible if all the S's are at the beginning of the word?

3. On a piano the white keys represent the following notes: C, D, E, F, G, A, B. How many tunes, seven notes in length, can be composed with these notes if:
   a) a note can be played only once?
   b) the notes can be repeated?
   c) the notes can be repeated and the tune begins and ends with a D?
   d) the tune consists of 3 D's, 2 B's and 2 A's.
4. There are three black beads and four white beads in a row. In how many ways can the beads be arranged if:
   a) same-coloured beads are treated as different beads?
   b) same-coloured beads are treated as identical beads?
5. There are eight balls on a table. Some are white and some are red. The white balls are all identical and the red balls are all identical. The balls are removed one at a time. In how many different orders can the balls be removed if:
   a) seven of the balls are red?
   b) three of the balls are red?
   c) there are four of each colour?
6. How many four-digit numbers can be formed with the digits 3, 4, 6 and 7 if:
   a) there can be repetition?
   b) each digit can only be used once?
   c) if the number is odd and repetition is allowed?
7. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 29J6 2. 29J7 3. 29J8 4. 29J9 5. 29JB 6. 29JC

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10.7 Application to probability problems EMCK5

When needing to determine the probability that an event occurs, and the total number of arrangements of the sample space, S, and the total number arrangements for the event, E, are very large, the techniques used earlier in this chapter may no longer be practical. In this case, the probability may be determined using the fundamental counting principle. The probability of the event, E, is the total number of arrangements of the event divided by the total number of arrangements of the sample space or $\frac{n(E)}{n(S)}$.

Worked example 18: Personal Identification Numbers (PINs)

QUESTION

Every client of a certain bank has a personal identification number (PIN) which consists of four randomly chosen digits from 0 to 9.

1. How many PINs can be made if digits can be repeated?
2. How many PINs can be made if digits cannot be repeated?
3. If a PIN is made by selecting four digits at random, and digits can be repeated, what is the probability that the PIN contains at least one eight?
4. If a PIN is made by selecting four digits at random, and digits cannot be repeated, what is the probability that the PIN contains at least one eight?

SOLUTION

1. If digits can be repeated: you have 10 digits to choose from and you have to choose four times, therefore the number of possible PINs $= 10^4 = 10\ 000$.

2. If digits cannot be repeated: you have 10 digits for your first choice, nine for your second, eight for your third and seven for your fourth. Therefore,

$$\text{the number of possible PINs } = 10 \times 9 \times 8 \times 7 = 5040$$

3. Let B be the event that at least one eight is chosen. Therefore the complement of B is the event that no eights are chosen.

   If no eights are chosen, there are only nine digits to choose from. Therefore, $n(\text{not } B) = 9^4 = 6561$.

   The total number of arrangements in the set, as calculated in Question 1, is $10\ 000$. Therefore:

$$\begin{aligned} P(B) &= 1 - P(\text{not } B) \\ &= 1 - \frac{n(\text{not } B)}{n(S)} \\ &= 1 - \frac{6561}{10\ 000} \\ &= 0{,}3439 \end{aligned}$$

4. Let B be the event that at least one eight is chosen. Therefore the complement of B, is the event that no eights are chosen.

   If no eights are chosen, there are only 9 then 8 then 7 then 6 digits to choose from as we cannot repeat a digit once it is chosen. Therefore,

$$n(\text{not } B) = 9 \times 8 \times 7 \times 6 = 3024$$

   The total number of arrangements in the set, as calculated in Question 1, is $10\ 000$. Therefore:

$$\begin{aligned} P(B) &= 1 - P(\text{not } B) \\ & 1 - \frac{n(\text{not } B)}{n(S)} \\ &= 1 - \frac{3024}{10\ 000} \\ &= 0{,}6976 \end{aligned}$$

Worked example 19: Number plates

QUESTION

The number plate on a car consists of any 3 letters of the alphabet (excluding the vowels and 'Q'), followed by any 3 digits (0 to 9). For a car chosen at random, what is the probability that the number plate starts with a 'Y' and ends with an odd digit?

SOLUTION

Step 1: Identify what events are counted

The number plate starts with a 'Y', so there is only 1 option for the first letter, and ends with an odd digit, so there are 5 options for the last digit $(1; 3; 5; 7; 9)$.

Step 2: Find the number of events

Use the counting principle. For the second and third letters, there are 20 possibilities (26 letters in the alphabet, minus 5 vowels and 'Q'). There are 10 possibilities for the first and second digits.

$$\text{Number of events } = 1 \times 20 \times 20 \times 10 \times 10 \times 5 = 200\ 000$$

Step 3: Find the total number of possible number plates

Use the counting principle. This time, the first letter and last digit can be anything.

$$\text{Total number of choices } = 20 \times 20 \times 20 \times 10 \times 10 \times 10 = 8\ 000\ 000$$

Step 4: Calculate the probability

The probability is the number of outcomes in the event, divided by the total number of outcomes in the sample space.

$$\text{Probability } = \frac{200\ 000}{8\ 000\ 000} = \frac{1}{40} = 0{,}025$$

Worked example 20: Probability of word arrangements

QUESTION

Refer to worked example 16 for context. If you take the word, 'BASSOON' and you randomly rearrange the letters, what is the probability that the word starts and ends with the same letter if repeated letters are treated as identical?

SOLUTION

If the word starts and ends with the same letter, there are a total number of 120 possible arrangements (from worked example 16). Let this event $= A$.

The total number of possible arrangements if repeated letters are treated as identical $= 1260$ (from worked example 16).

Therefore, the probability of an arrangement beginning and ending with the same letter

$$= \frac{n(A)}{n(S)} = \frac{120}{1260} = 0{,}1$$

Exercise 10 – 8: Solving probability problems using the fundamental counting principle

1. A music group plans a concert tour in South Africa. They will perform in Cape Town, Port Elizabeth, Pretoria, Johannesburg, Bloemfontein, Durban and East London.
   a) In how many different orders can they plan their tour if there are no restrictions?
   b) In how many different orders can they plan their tour if their tour begins in Cape Town and ends in Durban?
   c) If the tour cities are chosen at random, what is the probability that their performances in Cape Town, Port Elizabeth, Durban and East London happen consecutively? Give your answer correct to 3 decimal places.
2. A certain restaurant has the following course options available for a three-course set menu:

STARTERS	MAINS	DESSERTS
Calamari salad	Fried chicken	Ice cream and chocolate sauce
Oysters	Crumbed lamb chops	Strawberries and cream
Fish in garlic sauce	Mutton Bobotie	Malva pudding with custard
	Chicken schnitzel	Pears in brandy sauce
	Vegetable lasagne	
	Chicken nuggets	

   a) How many different set menus are possible?
   b) What is the probability that a set menu includes a chicken course?
3. Eight different pairs of jeans and 5 different shirts hang on a rail.
   a) In how many different ways can the clothes be arranged on the rail?
   b) In how many ways can the clothing be arranged if all the jeans hang together and all the shirts hang together?
   c) What is the probability, correct to three decimal places, of the clothing being arranged on the rail with a shirt at one end and a pair of jeans at the other?
4. A photographer places eight chairs in a row in his studio in order to take a photograph of the debating team. The team consists of three boys and five girls.
   a) In how many ways can the debating team be seated?
   b) What is the probability that a particular boy and a particular girl sit next to each other?
5. If the letters of the word 'COMMITTEE' are randomly arranged, what is the probability that the letter arrangements start and end with the same letter?
6. Four different Mathematics books, three different Economics books and two different Geography books are arranged on a shelf. What is the probability that all the books of the same subject are arranged next to each other?
7. A number plate is made up of three letters of the alphabet (excluding F and S) followed by three digits from 0 to 9. The numbers and letters can be repeated. Calculate the probability that a randomly chosen number plate:
   a) starts with the letter D and ends with the digit 3.
   b) has precisely one D.
   c) contains at least one 5.

8. In the 13-digit identification (ID) numbers of South African citizens:
    - The first six numbers are the birth date of the person in YYMMDD format.
    - The next four digits indicate gender, with 5000 and above being male and 0001 to 4999 being female.
    - The next number is the country ID; 0 is South Africa and 1 is not.
    - The second last number used to be a racial identifier but it is now 8 for everybody.
    - The last number is a control digit, which verifies the rest of the number.

    Assume that the control digit is a randomly generated digit from 0 to 9 and ignore the fact that leap years have an extra day.

    a) Calculate the total number of possible ID numbers.
    b) Calculate the probability that a randomly generated ID number is of a South African male born during the 1980s. Write your answer correct to two decimal places.
9. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 29JD 2. 29JF 3. 29JG 4. 29JH 5. 29JJ 6. 29JK
7. 29JM 8. 29JN

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10.8 Summary EMCK6

- The **addition rule** (also called the sum rule) for any 2 events, A and B is

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

This rule relates the probabilities of 2 events with the probabilities of their union and intersection.

- The **addition rule for 2 mutually exclusive events** is

$$P(A \text{ or } B) = P(A) + P(B)$$

This rule is a special case of the previous rule. Because the events are mutually exclusive, $P(A \text{ and } B) = 0$.

- The **complementary rule** is

$$P(\text{not } A) = 1 - P(A)$$

This rule is a special case of the previous rule. Since A and (not A) are complementary, $P(A \text{ or } (\text{not } A)) = 1$.

- The **product rule** for independent events A and B is:

$$P(A \text{ and } B) = P(A) \times P(B)$$

If two events A and B are **dependent** then:

$$P(A \text{ and } B) \neq P(A) \times P(B)$$

- **Venn diagrams** are used to show how events are related to one another. A Venn diagram can be very helpful when doing calculations with probabilities. In a Venn diagram each event is represented by a shape, often a circle or a rectangle. The region inside the shape represents the outcomes included in the event and the region outside the shape represents the outcomes that are not in the event.

- **Tree diagrams** are useful for organising and visualising the different possible outcomes of a sequence of events. Each branch in the tree shows an outcome of an event, along with the probability of that outcome. For each possible outcome of the first event, we draw a line where we write down the probability of that outcome and the state of the world if that outcome happened. Then, for each possible outcome of the second event we do the same thing. The probability of a sequence of outcomes is calculated as the product of the probabilities along the branches of the sequence.

- **Two-way contingency tables** are a tool for keeping a record of the counts or percentages in a probability problem. Two-way contingency tables are especially helpful for figuring out whether events are dependent or independent.

- The **fundamental counting principle** states that if there are $n(A)$ outcomes for event A and $n(B)$ outcomes for event B, then there are $n(A) \times n(B)$ different possible outcomes for both events.

- When you have n objects to choose from and you choose from them r times, if the number of choices remains the same after each choice, then the total number of possibilities is
$$n \times n \times n \ldots \times n \ \ (r \text{ times}) = n^r$$

- The number of arrangements of n different objects is
$$n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1 = n!$$

- For a set of n objects, of which there are k subsets with repeated objects i.e. n_1 are the same, n_2 are the same, $\ldots, n_k$ are the same, the number of arrangements are
$$\frac{n!}{n_1! \times n_2! \ldots n_k!}$$

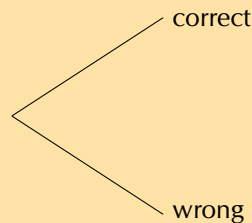
Exercise 10 – 9: End of chapter exercises

1. An ATM card has a four-digit PIN. The four digits can be repeated and each of them can be chosen from the digits 0 to 9.
   a) What is the total number of possible PINs?
   b) What is the probability of guessing the first digit correctly?
   c) What is the probability of guessing the second digit correctly?
   d) If your ATM card is stolen, what is the probability, correct to four decimal places, of a thief guessing all four digits correctly on his first guess?
   e) After three incorrect PIN attempts, an ATM card is blocked from being used. If your ATM card is stolen, what is the probability, correct to four decimal places, of a thief blocking the card? Assume the thief enters a different PIN each time.

2. The LOTTO rules state the following:

   - Six numbers are drawn from the numbers 1 to 49 - this is called a 'draw'.
   - Numbers are not replaced once drawn, so you cannot have the same number more than once.
   - The order of the drawn numbers does not matter.

   You decide to buy one LOTTO ticket consisting of 6 numbers.

   a) How many different possible LOTTO draws are there? Write your answer in scientific notation, rounding to two digits after the decimal point.
   b) Complete the tree diagram below after the first two LOTTO numbers have been drawn showing the possible outcomes and probabilities of the numbers on your ticket.

   c) What is the probability of getting the first number drawn correctly?
   d) What is the probability of getting the second number drawn correctly if you get the first number correct?
   e) What is the probability of getting the second number drawn correct if you do not get the first number correctly?
   f) What is the probability of getting the second number drawn correct?
   g) What is the probability of getting all 6 LOTTO numbers correct? Write your answer in scientific notation, rounding to two digits after the decimal point.

3. The population statistics of South Africa show that 55% of all babies born are female. Calculate the probability that a couple planning to have children will have a boy followed by a girl and then a boy. Assume that each birth is an independent event. Write your answer as a percentage, correct to two decimal places.
4. Fezile and Vuzi write a Mathematics test. The probability that Fezile will pass the test is $0{,}8$. The probability that Vuzi will pass the test is $0{,}75$. What is the probability that only one of them will pass the test?
5. Landline telephone numbers are 10 digits long. Numbers begin with a zero followed by 9 digits chosen from the digits 0 to 9. Repetitions are allowed.

   a) How many different phone numbers are possible?
   b) The first three digits of a number form an area code. The area code for Cape Town is 021. How many different phone numbers are available in the Cape Town area?
   c) What is the probability of the second digit being an even number?
   d) Ignoring the first digit, what is the probability of a phone number consisting of only odd digits? Write your answer correct to three decimal places.

6. Take the word 'POSSIBILITY'.

   a) In how many way can the letters be arranged if repeated letters are considered identical?
   b) What is the probability that a randomly generated arrangement of the letters will begin with three I's? Write your answer as a fraction.

7. The code to a safe consists of 10 digits chosen from the digits 0 to 9. None of the digits are repeated. Determine the probability of a code where the first digit is odd and none of the first three digits may be a zero. Write your answer as a percentage, correct to two decimal places.

8. Four different red books and three different blue books are to be arranged on a shelf. What is the probability that all the red books and all the blue books stand together on the shelf?

9. The probability that Thandiswa will go dancing on a Saturday night (event D) is $0{,}6$ and the probability that she will go watch a movie is $0{,}3$ (event M). Determine the probability that she will:

   a) go dancing and watch a movie if D and M are independent.
   b) go dancing or watch a movie if D and M are mutually exclusive.
   c) go dancing and watch a movie if $P(D \text{ or } M) = 0{,}7$.
   d) not go dancing or go to a movie if $P(D \text{ and } M) = 0{,}8$.

10. Three boys and four girls sit in a row.

    a) In how many ways can they sit in the row?
    b) What is the probability that they sit in alternating gender positions?

11. The number plate on a car consists of any 3 letters of the alphabet (excluding the vowels, J and Q), followed by any 3 digits from 0 to 9. For a car chosen at random, what is the probability that the number plate starts with a Y and ends with an odd digit? Write your answer as a fraction.

12. There are four black balls and y yellow balls in a bag. Thandi takes out a ball, notes its colour and then puts it back in the bag. She then takes out another ball and also notes its colour. If the probability that both balls have the same colour is $\frac{5}{8}$, determine the value of y.

13. A rare kidney disease affects only 1 in 1000 people and the test for this disease has a 99% accuracy rate.

    a) Draw a two-way contingency table showing the results if $100\ 000$ of the general population are tested.
    b) Calculate the probability that a person who tests positive for this rare kidney disease is sick with the disease, correct to two decimal places.

14. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 29JP 2. 29JQ 3. 29JR 4. 29JS 5. 29JT 6. 29JV 7. 29JW
8. 29JX 9. 29JY 10. 29JZ 11. 29K2 12. 29K3 13. 29GT

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Solutions to exercises

1 Sequences and series

Exercise 1 – 1: Arithmetic sequences

1. $26; 30; 34$
2. $-19; -22; -25$
3. $5; 7; 9$
4. $65; 76; 87$
5. $a+5b; a+7b; a+9b$
6. $\frac{3}{2}; 2; \frac{5}{2}$

Exercise 1 – 2: Arithmetic Sequences

1. a) 1
   b) $T_n = 8{,}5 - 1{,}5n$
   c) $T_{21} = -23$
2. a) Yes
   b) 218
   c) $T_{81} = 322$
   d) No
3. a) $T_1 = 5; T_2 = 3; T_3 = 1; T_{10} = -13$
   b)
4. $T_n = \frac{1}{2}n - 1$
7. a) $4; 7; 10; 13$
   b) $d = 3$
   c) $T_n = 3n + 1$
   d) $T_{25} = 76$
   e) $T_{36} = 109$
8. a) $p = 7; q = 7; r = 35; s = 2n + 1$
   b) 801 stitches
9. $p = \frac{1}{2}; T_{15} = 35\frac{1}{2}$
10. $x = 5a - 6$
11. $3s - t; s; -s + t; -3s + 2t; -5s + 3t; -7s + 4t; -9s + 5t; -11s + 6t; -13s + 7t;$

Exercise 1 – 3: Quadratic sequences

1. a) Quadratic sequence
   b) Linear sequence
   c) Quadratic sequence
   d) Linear sequence
   e) Quadratic sequence
   f) Neither
   g) Quadratic sequence
   h) Quadratic sequence
2. $b = 0$ and $c = -2$
3. a) Yes
   b) $-7a^2$
   c) $T_{100} = -197a^2$
4. $b = 2$ and $c = -1$
5. $4; 14; 34; 64; 104; 154$
6. a) -1
   b) 7

Exercise 1 – 4: Constant ratio of a geometric sequence

1. $r = 2$ next terms: $40; 80; 160$
2. $r = 2$ next terms: $\frac{1}{16}; \frac{1}{32}; \frac{1}{64}$
3. $r = 0{,}1$ next terms: $0{,}007; 0{,}0007; 0{,}00007$
4. $r = 3p$ next terms: $27p^4; 81p^5; 243p^6$
5. $r = -10$ next terms: $3000; -30\,000; 300\,000; -3\,000\,000$

Exercise 1 – 5: General term of a geometric sequence

1. $T_n = 5(2)^{n-1}$
2. $T_n =)\frac{1}{2}/^{n}$
3. $T_n = 7(0{,}1)^{n-1}$
4. $T_n = p(3p)^{n-1}$
5. $T_n = -3(-10)^{n-1}$

Exercise 1 – 6: Mixed exercises

1. a) $6; 2; \frac{2}{3} \ldots$
   b) Geometric: $r = \frac{1}{2}$
2. $k = 4; m = 10$
3. a) $r = 2$
   b) $a = \frac{1}{4}$
   c) $T_n = \frac{2^n}{8}$
4. a) $r = 2$ and $a = \frac{1}{4}$
   b) $y = \frac{1}{4}$ and $x = \frac{1}{2}$
   c) $T_5 = 4$
5. $T_4 = \sin^3 \theta$ and $T_5 = \sin^4 \theta$
6. Arithmetic: $5; \frac{1}{4}; -\frac{9}{2}; \ldots$ or $5; 25; 45; \ldots$
   Geometric: $\frac{1}{4}; -\frac{9}{2}; 81; \ldots$ or $25; -45; 81; \ldots$
7. a) $4x^3y$
   b) $r = \frac{2x}{y}$
8. $-1; \frac{1}{3}; -\frac{1}{9}; \frac{1}{27}; -\frac{1}{81}$ or $-1; -\frac{1}{3}; -\frac{1}{9}; -\frac{1}{27}; -\frac{1}{81}$

Exercise 1 – 7: Sigma notation

1. a) $2 + 2 + 2 + 2 = 8$
   b) -1
   c) 34
2. a) 0
   b) $8 + 8 + 8 + 8 = 32$
   c) $15a$
3. a) $a = 4$
   b) $a = \frac{15}{16}$
4. $\sum_{n=1}^{4})3^{n-3}/$
5. $\sum_{i=1}^{25} (18 - 7n)$
6. $\sum_{k=1}^{1000} (2n - 1)$
   $\sum_{k=0}^{999} (2n + 1)$

Exercise 1 – 8: Sum of an arithmetic series

1. $k = 4$
2. a) $n = 10$
   b) $T_6 = 46$
3. a) $n = 13$ or $n = 50$
4. $S_{51} = 4335$
5. $S_{300} = 135750$
6. $\frac{5}{3}$
7. a) $d = 5$ and $a = -9$
   b) $T_{100} = 486$
8. a) $S_9 = 324$
   b) 260
9. $n = 15$
10. $d = 10$
11. $n = 20$

Exercise 1 – 9: Sum of a geometric series

2. a) -2187
   b) -1640
3. $S_4 = 45$
4. $S_{11} = \frac{6141}{512}$
6. $a = 5; r = 2; S_7 = 635$
7. a) $4; 2; 1$
   b) $n = 9$
8. $r = \frac{3}{2}, a = \frac{32}{9}, T_2 = \frac{16}{3}$

Exercise 1 – 10: Convergent and divergent series

1. Arithmetic series: $S_1 = 2; S_2 = 6; S_{10} = 110; S_{100} = 10100$. Divergent.
2. Arithmetic series: $S_1 = -1; S_2 = -3; S_{10} = -55; S_{100} = -5050$. Divergent.
3. Geometric series: $S_1 = \frac{2}{3}; S_2 = \frac{10}{9}; S_{10} = 1{,}965\ldots; S_{100} = 2{,}00\ldots$. Convergent.
4. Geometric series: $S_1 = 2; S_2 = 6; S_{10} = 2046; S_{100} = 2{,}5 \times 10^{30}$. Divergent.

Exercise 1 – 11: Sum to infinity

1. $S_\infty = \frac{2}{3}$
2. $S_\infty = \frac{9}{2}$
3. $-4 < x < 2$
4. $a = \frac{5}{3}, r = \frac{3}{5}$
6.
7. $p = \frac{2}{3}$

Exercise 1 – 12: End of chapter exercises

1. Infinite arithmetic series
2. a) $S_7 = 28$
   b) $S_{n-1} \frac{n(n-1)}{2}$
3. $S_5 = \frac{121}{2187}$
4. a) $x = 3$
   b) $S_\infty = 8$
5. $\sum_{n=1}^{20} 6) \frac{1}{2}^{n-1}$
6. $S_\infty = 15$
7. a) R 250
   b) 30 years
8. $S_\infty = \frac{16}{3}$
9. $n = 10$
10. 150 mm
11. $n = 10$
12. a) R 708,62
    b) R 8553,71
13. a) R 524 288
    b) R 1 048 575
14. $9; 6; 4; \ldots$
15. a) $p = 10$
    c) $T_{10} = 119{,}2$
16. $S_\infty = 162$
17. 34
18. $T_2 = 12$
19. a) Converge: $S_\infty = -\frac{1}{3}$
    b) Diverge: $r = -3$
20. $S_\infty = \frac{5}{3}$
21. $T_{10} = 20$
22. 1 998 953
23. a) 25
    b) Letter: s
24. $\frac{26}{45}$
25. a) $-2 < x < 0$
    b) $f(-\frac{1}{2}) = \frac{2}{3}$
26. $S_n = \frac{a^2(1-a^{2n})}{(1-a^2)}$
27. $T_n = 59$
28. $n = 9$

2 Functions

Exercise 2 – 1: Revision

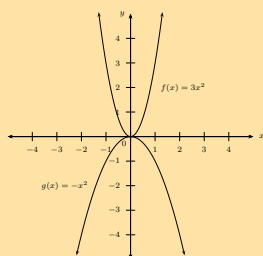
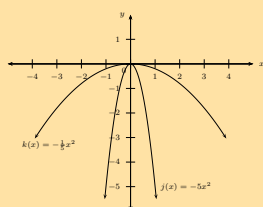
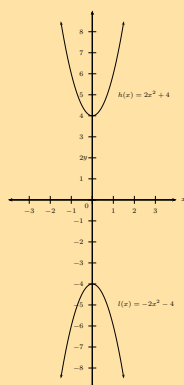
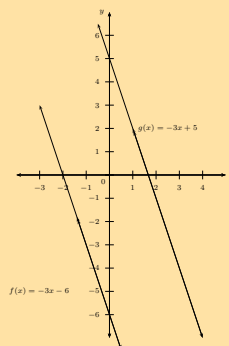
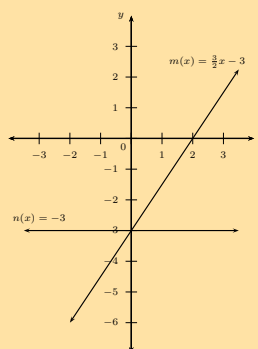
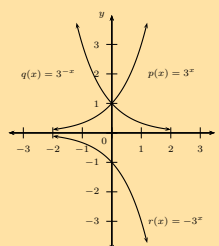
1. a)

b)

c)

2.

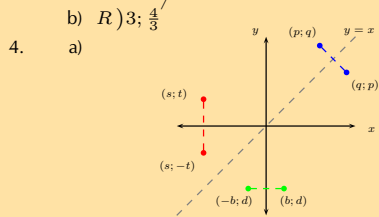
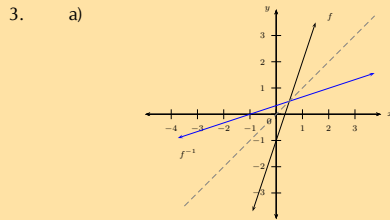
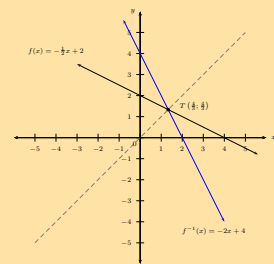
g(x) = −3x + 5

f(x) = −3x − 6

3.

4. a)

Exercise 2 – 2: Identifying functions

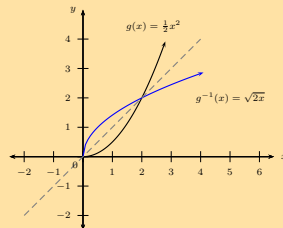
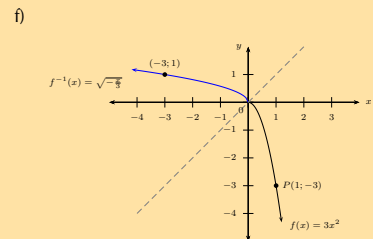
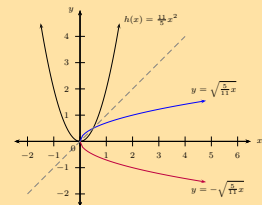
1. a) Yes
   b) Yes
   c) Yes
   d) No
   e) Yes
   f) No
   g) Yes
   h) No
2. a) One-to-one relation: therefore is a function.
   b) One-to-one relation: therefore is a function.
   c) One-to-one relation: therefore is a function.
   d) One-to-many relation: therefore is not a function.
   e) Many-to-one relation: therefore is a function.
3. $d = -500u + 23\ 000$

Exercise 2 – 3: Inverse of the function $y = ax + q$

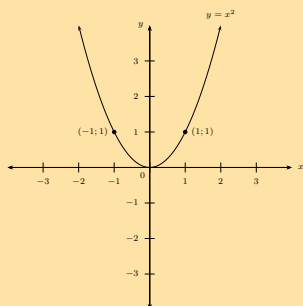
1. $f^{-1}(x) = \frac{1}{5}x - \frac{4}{5}$
2. a) It is a function.
   b) Domain $\{x : x \in \mathbb{R}\}$, Range $\{y : y \in \mathbb{R}\}$
   c) $f^{-1}(x) = -\frac{1}{3}x - \frac{7}{3}$
3. a)

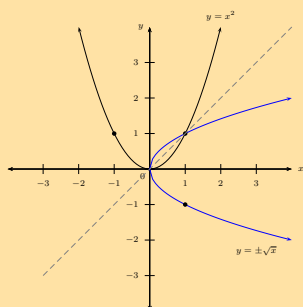
   b) $R\,)3; \frac{4}{3}/$
4. a)

   b) No
5. a) $f(x) = -\frac{1}{2}x + 2$
   b) $f(x) : (4;0), (0;2) f^{-1}(x) : (2;0)$ and $(0;4)$
   c) $T\,)\frac{4}{3}; \frac{4}{3}/$
   d)

   e) Decreasing function

Exercise 2 – 4: Inverses - domain, range, intercepts, restrictions

1. a) $y = \pm\sqrt{\frac{4}{3}x} \quad (x \geq 0)$
   b) $y = \pm\sqrt{\frac{1}{2}x} \quad (x \geq 0)$
   c) $y = \pm\sqrt{-5x} \quad (x \leq 0)$
   d) $y = \pm\sqrt{4x} \quad (x \geq 0)$
2. a) $g^{-1}(x) = \sqrt{2x} \quad (x \geq 0)$
   b)

   c) Yes
   d) For g :Domain: $x \geq 0$ Range: $y \geq 0$
      For g^{-1} :Domain: $x \geq 0$ Range: $y \geq 0$
   e) $(0;0)$ or $2;2$
3. a) $f(x) = -3x^2 \quad (x \geq 0)$
   b) f : domain $x \geq 0$, range $y \leq 0$
   c) $(-3;1)$
   d) $f^{-1}(x) = \sqrt{-\frac{1}{3}x}, \quad (x \leq 0)$
   e) f^{-1} : domain $x \leq 0$, range $y \geq 0$
   f)

4. a) $y = \pm\sqrt{\frac{5}{11}x} \quad (x \geq 0)$
   b)
   c) $x \geq 0$ or $x \leq 0$
5. a) $a = -\frac{1}{2}$, $c = 0$ and $m = -\frac{1}{4}$
   b) g : domain: $x \in \mathbb{R}$, range: $y \in \mathbb{R}$ and f^{-1} : domain: $x \geq 0$, range: $y \leq 0$
   c) $x > 4$
   d) $y = 4x^2 \quad (y \geq 0)$
   e) $(0;0)$ and $)-\frac{1}{16}; \frac{1}{64}/$

Exercise 2 – 5: Inverses - average gradient, increasing and decreasing functions

1. a)

   b) $y = \pm\sqrt{x} \quad (x \geq 0)$

   c)

   d) No

   e) $Q(4; 2)$

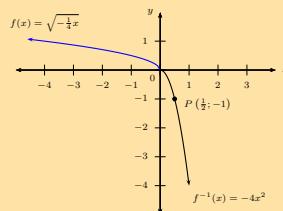
   f) Ave. grad.$_{OP} = 2$ and ave. grad.$_{OQ} = \frac{1}{2}$

2. a) $f^{-1}(x) = -4x^2$

   b) Domain: $x \geq 0$ and Range: $y \leq 0$

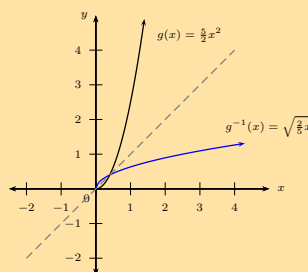
   c) $f(x) = \sqrt{-\frac{1}{4}x} \quad (x \leq 0)$

   d) Domain: $\{x : x \leq 0, x \in \mathbb{R}\}$, Range: $\{y : y \geq 0, y \in \mathbb{R}\}$

   e)

   f) Decreasing function

3. a) $g^{-1}(x) = \sqrt{\frac{2}{5}x} \quad (x \geq 0)$

   b) $(0; 0)$ and $\left(\frac{2}{5}; \frac{2}{5}\right)$

   c)

   d) Both are increasing functions

   e) 1

Exercise 2 – 6: Finding the inverse of $y = b^x$

1. a) $4 = \log_2 16$
   b) $-5 = \log_3 \left(\frac{1}{243}\right)$
   c) $3 = \log_{1,7} (4{,}913)$
   d) $x = \log_2 y$
   e) $\log_4 q = 5$
   f) $\log_y 4 = g$
   g) $\log_{(x-4)} 9 = p$
   h) $\log_m 3 = a + 4$
2. a) $2^5 = 32$
   b) $10^{-3} = \frac{1}{1000}$
   c) $10^{-1} = 0{,}1$
   d) $d^b = c$
   e) $5^0 = 1$
   f) $3^{-4} = \frac{1}{81}$
   g) 2
   h) -4

Exercise 2 – 7: Applying the logarithmic law: $\log_a x^b = b \log_a x$

1. $10 \log_8 10$
2. $y \log_{16} x$
3. $\frac{\log_3 5}{2}$
4. $z \log_z y$
5. $\frac{1}{x}$
6. q
7. $\frac{3}{4}$
8. -1
9. 15
10. 8
11. 4
12. -3

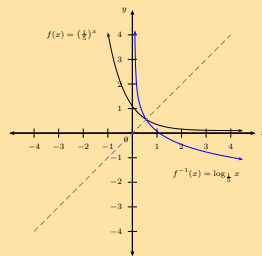
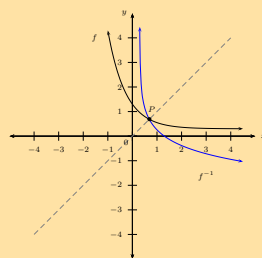
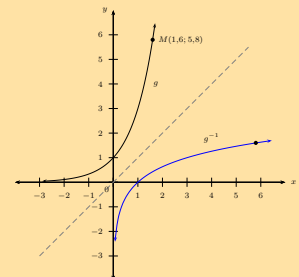
Exercise 2 – 8: Applying the logarithmic law: $\log_a x = \frac{\log_b x}{\log_b a}$

1. a) $\frac{\log_8 4}{\log_8 2}$
   b) $\frac{\log_2 14}{\log_2 10}$
   c) $\frac{\log_2 9-1}{\log_2 10}$
   d) $\frac{1}{\log_8 2}$
   e) $\frac{1}{\log_x y}$
   f) $\frac{1+\log_2 x}{\log_2 10}$
2. a) 1
   b) $\frac{2}{\log 5}$
3. a) $1{,}58$
   b) $6{,}92$

Exercise 2 – 9: Logarithms using a calculator

1. a) $0{,}477$
   b) $1{,}477$
   c) $2{,}477$
   d) $-0{,}180$
   e) $-0{,}602$
   f) $2{,}930$
   g) no value
   h) $1{,}262$
   i) -2
   j) $3{,}907$
   k) $1{,}661$
   l) $-2{,}585$
2. a) $3{,}98$
   b) $0{,}01$
   c) $63{,}10$
   d) $100\ 000$
   e) $0{,}32$
   f) $1{,}19$
   g) $2{,}51$
   h) $0{,}06$
   i) $1{,}19$
   j) $0{,}85$
   k) $0{,}06$
   l) $1{,}79$

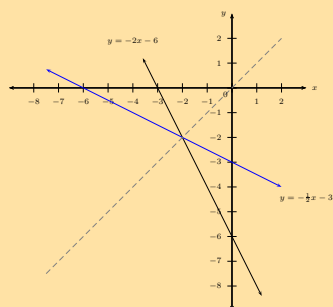
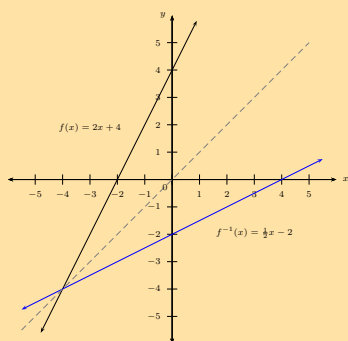
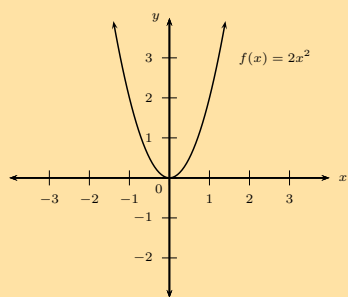
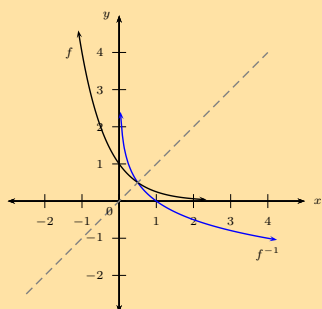
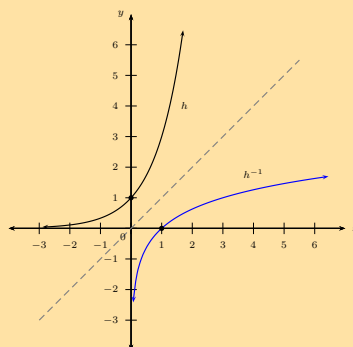
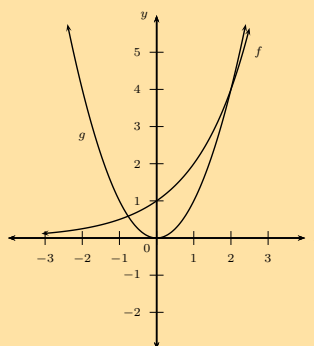
Exercise 2 – 10: Graphs and inverses of $y = \log_b x$

1. a)

   b) $f : (0; 1)$ and $f^{-1} : (1; 0)$
   c)

   d)
2. a) $t = 3$
   b) $g^{-1}(x) = \log_3 x \quad (x > 0)$
   c)

   d) $N(5,8; 1,6)$

Exercise 2 – 11: Applications of logarithms

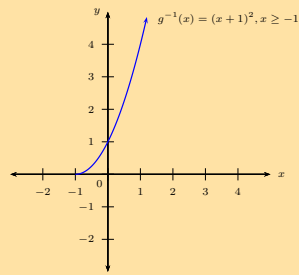
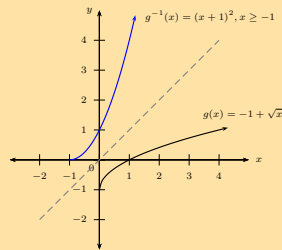
1. 57 years
2. a) Growth $= 36 \times 2^n$
   b) After 14 months

Exercise 2 – 12: End of chapter exercises

1. a) $h(x) = -2x - 6$
   b) $h^{-1}(x) = -\frac{x}{2} - 3$
   c)

   d) $S(-2; -2)$
   e) x-coordinate is equal to the y-coordinate.
2. a) $f(x) = \frac{1}{2}x - 2$
   b)

   c) Increasing
3. a)

   b) $y = \pm\sqrt{\frac{x}{2}}, \quad x \geq 0$. Domain: $\{x : x \geq 0, x \in \mathbb{R}\}$, Range: $\{y : y \in \mathbb{R}\}$.
4. a)

   b) Yes
   c) $y = \log_{\frac{1}{4}} x$ or $y = -\log_4 x$
   d) $P = \frac{1}{2}$, since the point lies on the line $y = x$.
   e) $G(x) = -\log_4 (x + 2)$ or $G(x) = \log_{\frac{1}{4}} (x + 2)$
   f) Vertical asymptote: $x = -2$
5. a) $h^{-1}(x) = \log_3 x$
   b) Domain: $\{x : x > 0, x \in \mathbb{R}\}$ and range: $\{y : y \in \mathbb{R}\}$.
   c)

   d) $0 < x < 1$
6. a)

   b) No
   c) Two
7. Graph 1: $y = 3^{-x}$
   Graph 2: $y = -\log_3 x$ or $y = \log_{\frac{1}{3}} x$
   Graph 3: $y = -3^x$
8. b) $a = \frac{1}{3}$
   c) $y = \log_3 x + 2$
   d) $y = \log_3 (x - 1)$
9. a) 10 years
   b) Graph C
10. a) $T_n = 100 \times 2^{n-1}$
    b) 204 800 retweets
    c) $1{,}75$

Exercise 2 – 13: Inverses (ENRICHMENT ONLY)

1. a) $g^{-1}(x) = (x+1)^2, \ x \geq -1$

   b)

   c)

   d) Yes

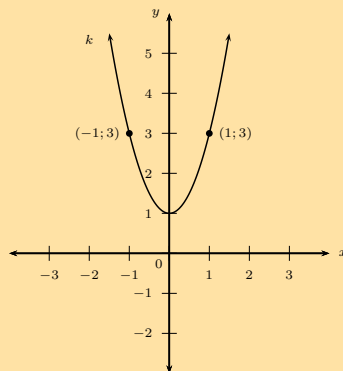
   e) Domain: $\{x : x \geq -1, x \in \mathbb{R}\}$, Range: $\{y : y \geq 0, y \in \mathbb{R}\}$.

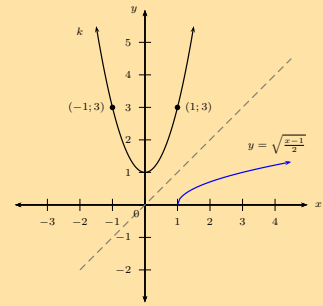
2. a) $y = -2(x-1)^2 + 3$

   b) Maximum value at $(1; 3)$

   c) Domain: $\{x : x \in \mathbb{R}\}$ and range: $\{y : y \leq 3, y \in \mathbb{R}\}$, Axis of symmetry: $x = 1$.

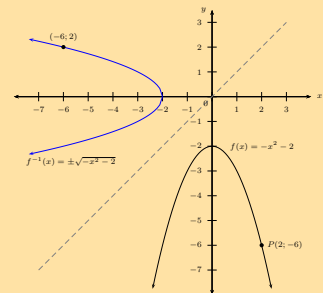
3. a) $q = 1$ or $q = -1$

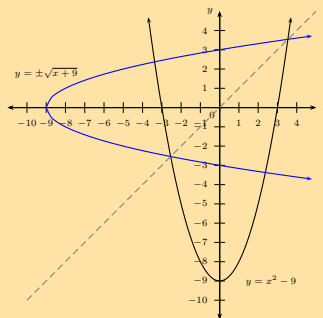
   b)

   c) $y = \pm\sqrt{\frac{x-1}{2}} \quad (x \geq 1)$

   d)

   e) $(3; 1)$

4. a) $f(x) = -x^2 - 2$

   b) $y = \pm\sqrt{-x-2} \quad (x \leq -2)$

   c)

5. a) $y = \pm\sqrt{x+9} \quad (x \geq -9)$

   b)

   c) No

   d)

Exercise 2 – 14: Applying logarithmic law: $\log_a xy = \log_a(x) + \log_a(y)$

1. a) $2\log_8 10$
   b) $1 + \log_2 7$
   c) $3 + \log_2 5$
   e) $1 + \log_2 x + \log_2 y$
   f) $\log_7$
2. a) $\log 30$
   b) 0
   c) $\log_3 12$
   d) $(\log x)(\log 10y)$
   e) Cannot be simplified.
   f) Cannot be simplified.
   g) $\log_a pq$
   h) $\log_a p \times \log_a q$
3. a) $\log xyz$
   b) $\log ab^2c^2d$
   c) $\log 2 + 3$
   d) 1

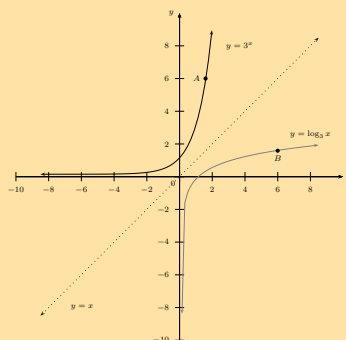
Exercise 2 – 15: Applying logarithmic law: $\log_a \frac{x}{y} = \log_a x - \log_a y$

1. a) $2 - \log 3$
   b) $\log_2 15 - 1$
   c) $\log_{16} x - \log_{16} y$
   d) Cannot be simplified.
   e) $1 - \log_5 8$
   f) $\log_x y - \log_x r$
2. a) $-\log 5$
   b) 2
   c) $\log_a \frac{p}{q}$
   d) Cannot be simplified
   e) Cannot be simplified
   f) $\log 3$
3. a) 1
   b) 2
4. Both methods are correct.

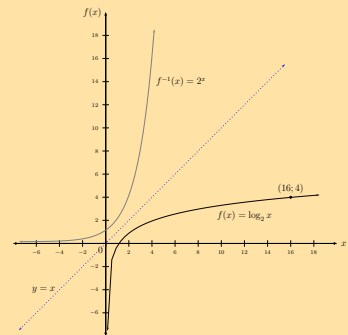
Exercise 2 – 16: Simplification of logarithms

1. 9
2. $\log \frac{18}{5}$
3. 2
4. 4

Exercise 2 – 17: Solving logarithmic equations

1. a) $1{,}09$
   b) $3{,}83$
   c) $5{,}66$
   d) $0{,}72$
   e) $-1{,}51$
   f) $65{,}94$
   g) $-0{,}55$
2. a) $y = \log_3 x$
   b) $1{,}6$
   c)

Exercise 2 – 18: Logarithms (ENRICHMENT ONLY)

1. a) False: $\log t + \log d = \log (t \times d)$
   b) False: $q = \log_p r$
   c) True
   d) False: $\log (A) - B$ cannot be simplified further.
   e) True
   f) False: $\log_k m = \frac{\log_p m}{\log_p k}$
   g) True
   h) True
   i) False: bases are different
   j) True
   k) False
   l) False
   m) False
   n) False
2. a) 1
   b) 0
   c) 2
   d) $\log 3$
3. a) 1,7
   b) 1,3
   c) 1,4
   d) $\frac{7}{3}$
   e) 5
4. b) 16
   c)

5. 1,68
6. a) $x < 10, (x \in \mathbb{Z})$
   b) $x < 11$

3 Finance

Exercise 3 – 1: Determining the period of an investment

1. Just over 3 years
2. Just over 5 years and 10 months
3. 3 years and 8 months ago
4. After 7 years
5. 5 years ago
6. 11
7. a) 71 years
   b) less than 5 years
8. 4,5 years

Exercise 3 – 2: Future value annuities

1. a) R 232 539,41
   b) R 210 000
   c) R 22 539,41
2. R 590,27
3. a) R 1 792 400,11
   b) R 382 800
4. R 2 923 321,08
5. a) R 510,85
   b) R 214 557
6. R 1123,28
7. a) 2 years and 211 days
   b) R 377,53

Exercise 3 – 3: Sinking funds

1. a) R 262 094,55
   b) No
2. a) R 104 384,58
   b) R 362 635,97
   c) R 4560,42
3. a) R 230,80
   b) R 1988

Exercise 3 – 4: Present value annuities

1. R 22 383,37
2. R 2420,00
3. R 6712,46
4. a) R 297,93
   b) R 32 176,44
5. a) 8 years
   b) R 2358,88
6. a) R 3917,91
   b) R 332 089,87
7. a) R 1627,05
   b) R 388 743,83
   c) R 18 530,10

Exercise 3 – 5: Analysing investment and loan options

1. a) option A
   b) R 1 937 512,76
2. Bank B

Exercise 3 – 6: End of chapter exercises

1. 6 years
2. R 102 130,80
3. a) R 394,50
   b) R 9796,00
4. 6 years
5. a) R 1 308 370,14
   b) R 468 000
6. a) R 134 243,45
   b) R 67 043,45
7. a) 5,5 years
   b) R 2465,87
8. a) R 1692,54
   b) R 230 651,34
   c) 1,4
9. R 8637,98
10. R 3895,68

4 Trigonometry

Exercise 4 – 1: Revision - reduction formulae, co-functions and identities

1. a) A
   b) A
   c) $\sqrt{1-A^2}$
   d) $-A$
   e) $\frac{A}{\sqrt{1-A^2}}$
2. a) $\sin^2\theta$
   b) $\cos^2\theta$
   c) $P+Q=1$ and $\frac{Q}{P}=\frac{1}{\tan^2\theta}$
3. $-\frac{1}{p}$
4. a) $\sqrt{3}$
   b) $\frac{1}{2}$
5. a) -1
   b) $\tan^2\theta$
   c) $-\cos^2\alpha$
   d) 0
6. a) $\sin\theta$
   b) $-\frac{1}{2}$
9. a) False
   b) False
   c) True
   d) False
   e) True
   f) True

Exercise 4 – 2: Compound angle formulae

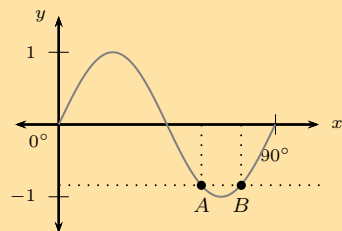
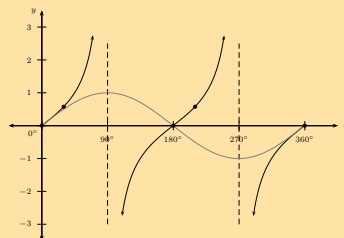
1. a) $\frac{5}{6}$
   b) $\frac{120}{169}$
   c) -1
2. a) $\frac{\sqrt{2}(\sqrt{3}+1)}{4}$
   b) $\frac{\sqrt{2}(1+\sqrt{3})}{4}$
   c) $\frac{\sqrt{2}(\sqrt{3}-1)}{4}$
   d) $2-\sqrt{3}$
   e) $\frac{1}{2}$
   f) 1
   g) $\frac{\sqrt{2}}{2}$
   h) $\frac{\sqrt{3}}{2}$
3. b) $\frac{\sqrt{6}}{2}$
4. $\sqrt{2}$
5. b) $\frac{2-\sqrt{3}}{4}$

Exercise 4 – 3: Double angle identities

1. a) $-\frac{7}{25}$
   b) $-\frac{24}{25}$
   c) $\frac{24}{7}$
2. a) $-t$
   b) $\sqrt{1-t^2}$
   c) t
   d) $2t^2 - 1$
   e) $-t$
   f) $2t^2 - 1$
3. b) $36{,}87°$ or $216{,}87°$
4. a) $\sqrt{\frac{2-\sqrt{2}}{4}}$
   b) $\sqrt{\frac{2-\sqrt{2}}{4}}$

Exercise 4 – 4: Solving trigonometric equations

1. a) $x = 16{,}06° + k \,.\, 180°$ or
   $x = 73{,}9° + k \,.\, 180°, k \in \mathbb{Z}$
   b) $y = 30° + k \,.\, 120°$ or
   $y = 90° + k \,.\, 360°, k \in \mathbb{Z}$
   c) $\alpha = 22{,}5° + k \,.\, 90°, k \in \mathbb{Z}$
   d) $p = k \,.\, 360°$ or
   $p = 36° + k \,.\, 72°, k \in \mathbb{Z}$
   e) $A = 45° + k \,.\, 180°$ or
   $A = 135° + k \,.\, 180°, k \in \mathbb{Z}$
   f) $x = 51{,}8° + k \,.\, 360°$ or
   $x = 308{,}2° + k \,.\, 360°, k \in \mathbb{Z}$
   g) $t = 0° + k \,.\, 180°, \quad k \in \mathbb{Z}$
   h) $x = 30° + k \,.\, 360°$
2. a) $x = 0°, 30°, 180°, 210°$ or $360°$
   b)

3. a) $A = 35{,}79° + k \,.\, 90°$ or
   $A = 31{,}72° + k \,.\, 90°, \quad k \in \mathbb{Z}$
   b) 10 solutions.
4. $A = -300°, -60°, 60°$ or $300°$
5. a) $x = 59{,}26° + k \,.\, 90°$ or
   $x = 75{,}74° + k \,.\, 90°, k \in \mathbb{Z}$.
   b)

$A(59{,}26°; -0{,}84)$, $B(75{,}74°; -0{,}84)$

7. $x = 60° + k \,.\, 360°$

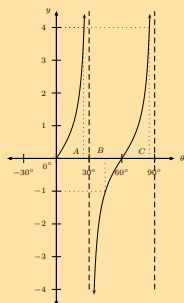
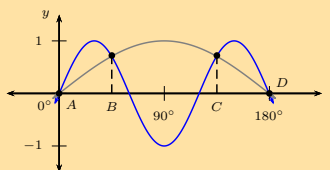
Exercise 4 – 5: Problems in two dimensions

1. a) $A\hat{O}C = 2\theta$
   b) $\cos\theta = \frac{AE}{BE}$
   c) $\sin\theta = \frac{CA}{AE}$
   d) $\sin 2\theta = \frac{CA}{AO}$
3. b) $CA^2 =$
   $DC^2 + DA^2 - 2(DC)(DA)\cos\hat{D}$ or
   $CA^2 =$
   $AB^2 + BC^2 - 2(AB)(BC)\cos\hat{B}$
   d) $CA^2 = 188{,}5$ units^2
   e) Area $ABCD = 54{,}0$ units^2
4. a) $AC = 69$ m
   b) $BD = 64$ m

Exercise 4 – 6: Problems in three dimensions

1. a) $BC = 2x\cos\alpha\tan\theta$
   b) $BC = 41{,}40$ m
2. a) $h = \frac{d\sin x\tan z}{\sin(x+y)}$
   b) 15 m
3. b) 16,2 m
   c) Approx. 6 levels
4. c) 1932,3 m^2
5. a) 229 m^2
   c) 148°

Exercise 4 – 7: End of chapter exercises

1. a) $\frac{1+\sqrt{3}}{2\sqrt{2}}$
   b) $\frac{\sqrt{3}-1}{2\sqrt{2}}$
   c) $2+\sqrt{3}$
   d) $\frac{1}{\sqrt{2}}$
   e) $\frac{1}{\sqrt{2}}$
2. $\cos 2\theta = -0{,}02$ and $\cos 4\theta = -0{,}9992$
3. a) $\frac{31}{49}$
   b) $-\frac{6\sqrt{40}}{31}$
4. $\frac{4}{5}$
5. $\frac{1}{2}$
6. $\theta \in \{-60°; -30°; 30°; 60°\}$
7. a) $\theta = 30° + k \,.\, 360°$ or $\theta = 150° + k \,.\, 360°, k \in \mathbb{Z}$
   b) $x = 30° + k \,.\, 360°$, $x = 150° + k \,.\, 360°, k \in \mathbb{Z}$, $x = 135° + k \,.\, 360°$ or $x = 225° + k \,.\, 360°, k \in \mathbb{Z}$
   c) $x = 10° + k \,.\, 360°$ or $x = 116{,}7° + k \,.\, 120°, k \in \mathbb{Z}$
   d) $\alpha = 11{,}3° + k \,.\, 180°$ or $\alpha = 135° + k \,.\, 180°, k \in \mathbb{Z}$
8. b) $\theta = 52{,}53°$ or $\theta = 127{,}47°$
10. b) Restricted values are: $y = 90° + k \,.\, 180°, k \in \mathbb{Z}$
11. a) $\theta = 45° + k \,.\, 60°$ or $\theta = 25{,}32° + k \,.\, 60°, k \in \mathbb{Z}$
    b) $\theta = 25{,}32°, 45°$, or $85{,}32°$
    c)

   d) $30° < \theta < 45°$
12. b) $x = -180°; -135°; -45°; 0°$ $= 45°; 135°; 180°; 225°; 315°; 360°$
    c)

13. a) $x = 30°; 270°; 150°$
    b)

14. b) 3
    c) $x = 360°$
    d) $x = 29°, 180°, 331°$
    e) $29° \leq x \leq 180°$ or $331° \leq x \leq 360°$
16. $f(\theta) = \frac{3}{2}\sin 2\theta$ and $g(\theta) = -\frac{3}{2}\tan\theta$
18. a) $\hat{U} = 108{,}6°$
    b) $\hat{S} = 71{,}4°$
    c) $R\hat{T}S = 54{,}9°$
19. c) 4 m
20. a) length = 8 cm, breadth = 6 cm, diagonal = 10 cm
    b) 7 cm
    c) $\frac{10}{7}$
21. a) $A\hat{B}C = 90°, A\hat{B}D = 90°$
    d) 8,4 m
22. b) 2 m

5 Polynomials

Exercise 5 – 1: Identifying polynomials

1. a) True
   b) True
   c) False: $f\left(\frac{1}{2}\right) = 0$
   d) True
   e) False: -1
   f) True
2. a) 4
   b) 3
   c) -9
   d) 6
3. a) Cubic polynomial
   b) Quadratic polynomial
   c) Not a polynomial
   d) Not a polynomial
   e) Cubic polynomial
   f) Linear polynomial
   g) Zero polynomial
   h) Polynomial; degree 7
   i) Cubic polynomial
   j) Not a polynomial

Exercise 5 – 2: Quadratic polynomials

1. a) $p = 0$ or $p = -2$
   b) $k = 4$ or $k = -9$
   c) $h = \pm 5$
   d) $x = -7$ or $x = -2$
   e) $y = 4k$ or $y = k$
2. a) $p = -5 - 3\sqrt{3}$ or $p = -5 + 3\sqrt{3}$
   b) $y = -3 \pm \sqrt{7}$
   c) No real solution
   d) $f = -8 \pm 3\sqrt{6}$
   e) $x = -1 \pm \sqrt{\frac{5}{3}}$
3. a) $m = 1$ or $m = -\frac{4}{3}$
   b) No real solution
   c) $y = 2 \pm \sqrt{2}$
   d) $f = \frac{1}{2}$ or $f = -2$
4. a) $(3p - 1)(9p^2 + 3p + 1)$
   b) $2\,)2 + \frac{1}{x}/\;\left(4 - \frac{2}{x} + \frac{1}{x^2}\right)$

Exercise 5 – 3: Cubic polynomials

1. a) $(p - 1)(p^2 + p + 1)$
   b) $(t + 3)(t^2 + 3t + 9)$
   c) $(4 - m)(16 + 4m + m^2)$
   d) $k(1 - 5k)(1 + 5k + 25k^2)$
   e) $)2a^2 - b^3/\;)4a^4 + 2a^2b^3 + b^6/$
   f) $(2 - p - q)(4 + 2p + 2q + p^2 + 2pq + q^2)$
2. a) $a(x) = (x + 1)(x^2 + x + 2) + 5$
   b) $a(x) = (x + 2)(-x^2 + 6x - 17) + 35$
   c) $a(x) = (x - 1)(2x^2 + 5x + 6)$
   d) $a(x) = (x - 1)(x^2 + 3x + 3) + 8$
   e) $a(x) = (x - 1)(x^3 + 3x^2 + 5) + 9$
   f) $a(x) = (x^2 - 2)(5x^2 + 3x + 16) + 7x + 34$
   g) $a(x) = (3x - 1)(x^2 + \frac{2}{3}) + \frac{5}{3}$
   h) $a(x) = (x + 2)(2x^4 - 4x^3 + 9x^2 - 15x + 30) - 64$
3. a) $f(x) = (x + 2)(x + 3) - 5$
   b) $f(x) = (x - 1)(x - 4) - 11$
   c) $f(x) = (x - 1)(2x^2 + 2x + 7) + 3$
   d) $f(x) = (x + 3)(x + 5) + 4$
   e) $f(x) = (x - 1)(x^2 + 3x + 4) - 6$
   f) $f(x) = (x + 3)(2x^2 + x - 1)$
   g) $f(x) = (2x - 1)\,(2x^2 + 3x + 1) - 1$
   h) $f(x) = (2x + 3)(x^2 - x + 4) + 10$

Exercise 5 – 4: Remainder theorem

1. a) 11
   b) 7
   c) -25
   d) 11
   e) -4
   f) -26
   g) 2
2. $t = 4$
3. $m = 7$
4. $k = -\frac{3}{2}$
5. $p = -5$
6. $b = -\frac{15}{2}$
7. $h = -6$
8. $m = 2 - n$
9. $p = -5, q = -11$

Exercise 5 – 5: Factorising cubic polynomials

1. -14
2. $a(x) = (x + 1)(x - 2)^2$
3. a) 0
   b) $f(x) = (x - 1)(2x - 1)(x + 2)$
4. $a(x) = (x - 1)(x + 5)(x - 3)$
5. a factor of $f(x)$

Exercise 5 – 6: Solving cubic equations

1. $x = -1$ or $x = 4$ or $x = -4$
2. $n = -2$ or $n = -4$ or $n = 5$
3. $y = -1$ or $y = 4$ or $y = -5$
4. $k = -2$ or $k = -3$ or $k = -4$
5. $x = -2$ or $x = 5$ or $x = -5$
6. $p = 3$ or $p = 2$ or $p = -5$
7. $x = -1$ or $x = 3$ or $x = 4$

Exercise 5 – 7: End of chapter exercises

1. $x = 1$ or $x = -3$
2. $y = -1$ or $y = -2$ or $y = 6$
3. $m = 1$ or $m = -2$ or $m = 2$
4. $x = -2$ or $x = \frac{3+\sqrt{21}}{2}$ or $x = \frac{3-\sqrt{21}}{2}$
5. $x = -1$ or $x = -\frac{1}{2}$ or $x = 3$
6. $x = -1$ or $x = -4$ or $x = 4$
7. b) $x = 2$ or $x = \frac{2}{3}$ or $x = 1$
8. $R = 1$
9. a) $m = 1$ or $m = \frac{1}{8}$ or $m = -2$
   b) $x = 0$ or $x = -3$
10. $R = 40$
11. $p = -\frac{12}{7}$ or $p = 1$
12. $t = 9$ and $Q(x) = x + 5$

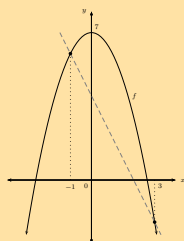
6 Differential calculus

Exercise 6 – 1: Limits

1. a) 0
   b) $\frac{1}{3}$
2. a) 4
   b) 7
   c) $\frac{37}{6}$
   d) Does not exist
   e) 0
   f) Does not exist
   g) 3
   h) 3
   i) $\frac{\sqrt{3}}{6}$

Exercise 6 – 2: Gradient at a point

1. a) -2
   b)

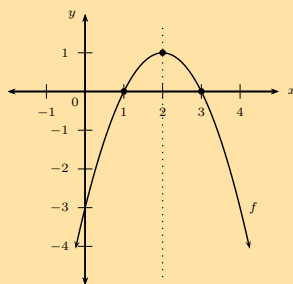
   c) $-2x$
2. $-\frac{3}{a^2}$
3. $y = x - 1$

Exercise 6 – 3: Differentiation from first principles

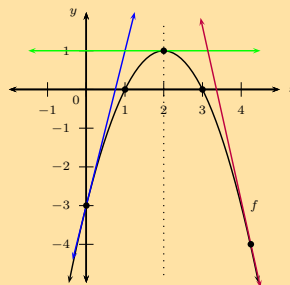
1. a) $\frac{-(x+h)^2-(-x^2)}{h}$
   b) $-2x$
2. $-4x + 3$
3. $\frac{-1}{(x-2)^2}$
4. $g'(3) = -30$
5. $8x - 4$
6. $k'(x) = 30x^2$
7. $f'(x) = nx^{n-1}$

Exercise 6 – 4: Rules for differentiation

1. a) $6x$
   b) 25
   c) 0
   d) $20x^4$
   e) $-\frac{32}{x^3}$
   f) 0
   g) $4x^3 - 12x$
   h) $2x + 1$
   i) $x^2 - 2x$
   j) $\frac{9}{2}x^{\frac{1}{2}} - 4$
   k) $2x + 7$
   l) $600x^2 - 200x + 40$
   m) $42x^2 - 28x + 7$
2. 1
3. $\frac{1}{2\sqrt{y}}$
4. $2z$
5. $2x - \frac{1}{\sqrt{x^3}} + \frac{3}{x^2}$
6. $\frac{3}{2}\sqrt{x} - \frac{1}{x^4}$
7. $3x^2 - 6 - \frac{9}{x^2}$
8. $\frac{1}{2}$
9. $6\theta^2 - 12 - \frac{18}{\theta^2}$
10. $\frac{5}{2}t^{\frac{3}{2}} + \frac{9}{2}3t^{\frac{1}{2}} + \frac{3}{2t^{\frac{1}{2}}} - \frac{1}{2t^{\frac{3}{2}}}$

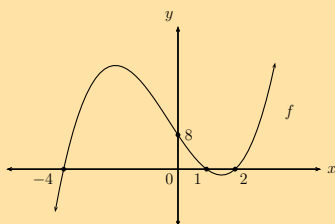
Exercise 6 – 5: Equation of a tangent to a curve

1. $y = 13x - 23$
2. a) $\left(-\frac{5}{6}; -\frac{13}{12}\right)$
   b) $(-3; -2)$
3. a) $(1; 1)$
   b) $\left(\frac{3}{4}; \frac{1}{4}\right)$
4. a)

b) $i) y = 4x - 3$; $ii) y = 1$; $iii) y = -4{,}5x + 15{,}0625$

c)

Exercise 6 – 6: Second derivative

1. a) 10
   b) $48x$
   c) 2
   d) $20x^3 - 6x$
   e) $6x - 2$
   f) $\frac{-60}{x^4}$
   g) $-\frac{1}{4\sqrt{x^3}} + 10$
2. $f'(x) = 20x + 15$; $f''(x) = 20$
3. $-\frac{4}{3\sqrt[3]{x^4}}$
4. a) $g'(x) = -6 + 24x - 24x^2$; $g''(x) = 24 - 48x$
   b) $g'(x)$: parabolic/quadratic; $g''(x)$: linear
   c) 0

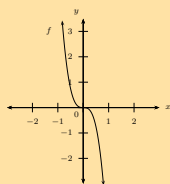
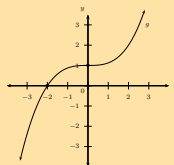
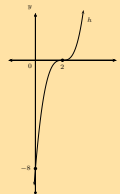
Exercise 6 – 7: Intercepts

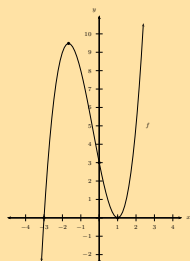
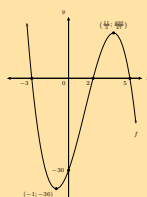
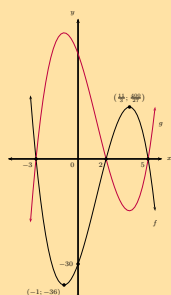
1. a) $(-4; 0)$, $(1; 0)$, $(2; 0)$ and $(0; 8)$
   b)

   c) Increasing
2. a) $(-5; 0)$, $(-3; 0)$, $(3; 0)$ and $(0; 45)$
   b) $(-1; 0)$, $\left(\frac{1}{4}; 0\right)$, $(2; 0)$ and $\left(0; \frac{1}{2}\right)$
   c) $(1; 0)$, $(\sqrt{12}; 0)$, $(-\sqrt{12}; 0)$ and $(0; 12)$
   d) $(0; 0)$, $(-4; 0)$ and $(4; 0)$
   e) $(0; 6)$, $(-1; 0)$, $(3 - \sqrt{3}; 0)$ and $(3 + \sqrt{3}; 0)$
3. $(-5; 0)$, $(0; 0)$ and $(2; 0)$

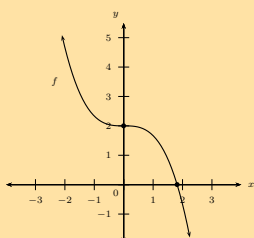
Exercise 6 – 8: Stationary points

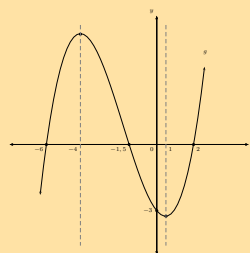
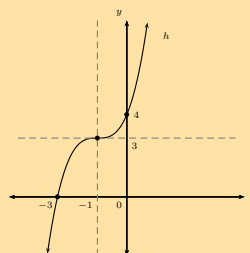
1. $\left(\frac{5}{2}; \frac{1}{4}\right)$
2. $x = -2$ or $x = 3$
3. a) $(1; 0)$
   b) (0;1) and $\left(\frac{10}{3}; -\frac{473}{27}\right)$
   c) None

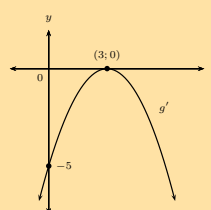
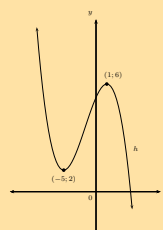
Exercise 6 – 9: Concavity and points of inflection

1.

2.

3.

Exercise 6 – 10: Mixed exercises on cubic graphs

1. a) $f(x) = (x-1)(x+3)(x-1)$
   b) $(0;3)$; $(1;\emptyset)$; $(-3;0)(1;0)$ and $)-\frac{5}{3};\frac{256}{27}$
   c)

2. a)

   b)

3. a) $f(x) = x^3 - 9x^2 + 24x - 20$
   b) $A(4;-4)$

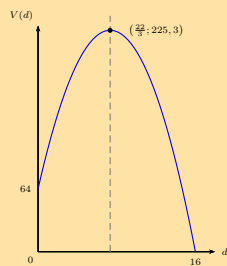
4. a)

   b) i.$x > \sqrt[3]{6}$; ii.$x \in R$, $x \neq 0$ and iii.$x > 0$

5. a)

   b)

6. a) 2 units
   b) 1 unit
   c) 5 units
   d) 12 units
   e) $C =)-\frac{2}{3};-\frac{400}{27}/$ and $F = (4;36)$
   f) 7 units
   g) 12
   h) $y = 15x - 15$

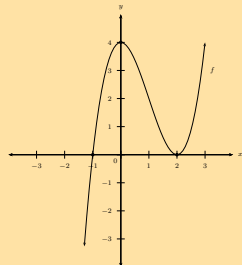
7.

8.

Exercise 6 – 11: Solving optimisation problems

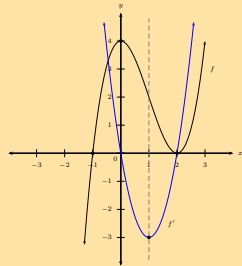
1. $\frac{20}{3}$ and $\frac{40}{3}$
2. b) $x = 10$ cm
3. 1 unit
4. $x = 5$ m and $y = 10$ m
5. b) $x \approx 7{,}9$ cm and $h =\approx 12{,}0$ cm

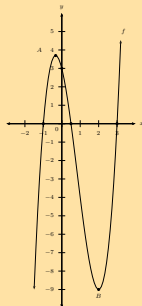
Exercise 6 – 12: Rates of change

1. a) -4 kℓ per day
   b) Decreasing
   c) 16 days
   d) $7\frac{1}{3}$ days
   e) 225,3 kℓ
   f)

2. a) 1 m
   b) 18 m.s^{-1}
   c) 9 m.s^{-1}
   d) 28 metres
   e) -6 m.s^{-2}
   f) 3 m.s^{-1}
   g) 0 m.s^{-1}
   h) 6,05 s
   i) $-18{,}3$ m.s^{-1}
3. $a = 6$ m.s^{-2}
4. a) $T'(t) = 4 - t$
   b) $(4; 10]$

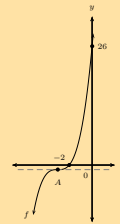
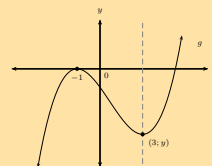
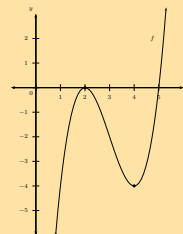
Exercise 6 – 13: End of chapter exercises

1. $f'(x) = -2x + 2$
2. $f'(x) = -\frac{1}{x^2}$
3. 3
4. a) $-3 - 10x$
   b) $8x - 2$
   c) $8x + \frac{1}{3x^2}$
   d) $\frac{5}{2\sqrt{x^3}}$
5. a) $f'(x) = 4x - 1$
   b) $(2; 6)$
6. $g'(2) = \frac{255}{8}$
7. a) $y = \frac{x}{2} + \frac{3}{2}$
   b) 9
8. a) $p'(t) = \frac{1}{5\sqrt[5]{t^2}}$
   b) $k'(n) = 6 + \frac{15}{n^2} + \frac{20}{n^3}$
9. $y = \frac{1}{2\sqrt{x}} - \frac{5}{x^2}$
10. a) $\frac{dy}{dx} = 3x^2$
    b) $\frac{dx}{dy} = \frac{1}{3\sqrt[3]{y^2}}$
11. a) 0
    c) $y = 3x(x - 2)$
    d)

e) $(-1; 0)$ and $(3; 4)$

f)

g) $f''(x) = 6x - 6$

12. a) $(-1; 0),)\frac{1}{2}; 0/$ and $(3; 0)$
    b) $)-\frac{1}{3}; \frac{100}{27}/$ and $(2; -9)$
    c)

d) $k = 3{,}7$ or $k = -9$

e) $y = -8x + 4$

13. a) $f(x) = x^3 + 9x^2 + 27x + 26$
    b) -1
    c)

14. a) $x = -1$ and $x = 3$
    b) $g'(0) = -2$
    c)

15. $h(x) = -x + 13$
16. b) $y = 9x - 23$
    c) $C(-1; -32)$
    d) $(-1; 0)$ and $(3; 0)$
17. a)

b) $y = -4$

c) $(3; -2)$

18. 2
19. a) 86 mm
    b) $-3{,}125$ mm per minute
    c) After 4 minutes
20. 24 m.s^{-2}
21. $10 + 5\sqrt{2}$ units
22. a) $H = \frac{1}{\pi r^2}$
    c) $d = 0{,}94$ m
23. a) $V = \frac{1}{12}(10\pi d^2 - \pi d^3)$
    b) $r = 3{,}34$ cm and $h = 3{,}34$ cm
    c) Maximum volume $= 38{,}79$ cm^3
24. a) $-11{,}5$ m per hour
    b) Decreasing
    c) 06h49

Exercise 7 – 1: Revision

1. a) $5\sqrt{10}$; $M =)\frac{1}{2}; \frac{7}{2}/$; $m = 3$; $y = 3x + 2$
   b) $\sqrt{306}$; $M =)\frac{5}{2}; \frac{3}{2}/$; $m = \frac{3}{5}$; $y = \frac{3}{5}x$
   c) $\sqrt{5h^2 - 20hk + 20k^2}$; $M = \left(\frac{h+2k}{2}; -3k\right)$; $m = -2$; $y = -2x + h - k$
   d) $\sqrt{104}$; $M = (1; 4)$; $m = 5$; $y = 5x - 1$
2. $A(7; 0)$
3. $r = -1$ or $r = 23$
4. a) $y = 2x + 3$
   b) $y = -x - 1$
   c) $y = \frac{1}{3}x - 2$
   d) $y = -4$
   e) $y = -4x + 1$
   f) $x = 5$
   g) $x = \frac{3}{4}$
   h) $y = 2px + 3$
   i) $y = 4x - \frac{3}{5}$
   j) $y = -2$
   k) $y = \frac{7}{2}x + 5$
   l) $y = -5x$

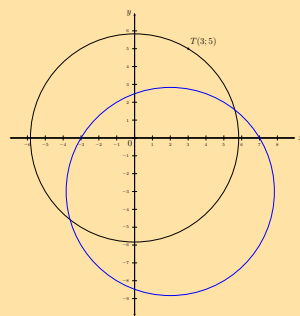
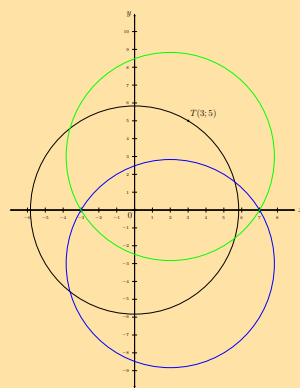
Exercise 7 – 2: Inclination of a straight line

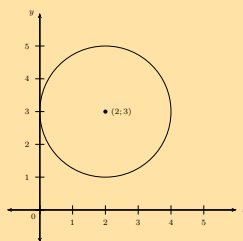
1. a) 36,9°
   b) 26,6°
   c) 45°
   d) Horizontal line
   e) 60°
   f) 68,2°
   g) 153,4°
   h) 56,3°
2. 86,6°
3. $y = -3x + 5$
4. $y = \frac{3}{2}x + 2$
5. $y = -5x - 1$
6. $y = x - 4$
7. a) $y = /2x + 7$
   b) $)\frac{7}{2}; 0/$
   c) $\theta = 116{,}6°$
   d) $m = \frac{1}{2}$
   e) $B\hat{A}C = 90°$
   f) $y = -2x/$
   g) $)\frac{1}{2}; -\frac{3}{2}/$
   h) $y = -2x - \frac{1}{2}$
8. a)
   b) $I\,(6; 9)$
9. a) $y = \frac{3}{2}x + 2$
   b) $T\hat{S}V = 45°$

Exercise 7 – 3: Equation of a circle with centre at the origin

1. a) $r = 4$, $\left(1; \sqrt{15}\right)$ and $\left(1; -\sqrt{15}\right)$
   b) $r = 10$, $\left(2; \sqrt{96}\right)$ and $\left(2; -\sqrt{96}\right)$
   c) $r = 3$, $\left(1; \sqrt{8}\right)$ and $\left(1; -\sqrt{8}\right)$
   d) $r = \sqrt{20}$, $(2; 4)$ and $(2; -4)$
   e) $r = \frac{3}{2}$, $\left(1; \frac{\sqrt{5}}{2}\right)$ and $\left(1; -\frac{\sqrt{5}}{2}\right)$
   f) $r = \frac{\sqrt{10}}{3}$, $)1; \frac{1}{3}/$ and $)1; -\frac{1}{3}/$
2. a) $x^2 + y^2 = 25$
   b) $x^2 + y^2 = 11$
   c) $x^2 + y^2 = 34$
   d) $x^2 + y^2 = \frac{25}{4}$
   e) $x^2 + y^2 = 225$
   f) $x^2 + y^2 = p^2 + 9q^2$
   g) $x^2 + y^2 = 1$
   h) $x^2 + y^2 = 29t^2$
   i) $x^2 + y^2 = \frac{49}{4}$
   j) $x^2 + y^2 = \frac{16}{9}$
3. a) Yes: $x^2 + y^2 = 8$
   b) No, cannot be written in the form $x^2 + y^2 = r^2$
   c) No, cannot be written in the form $x^2 + y^2 = r^2$.
   d) Yes: $x^2 + y^2 = \sqrt{6}$
   e) Yes: $x^2 + y^2 = 11$
   f) No, cannot be written in the form $x^2 + y^2 = r^2$.
   g) Yes: $x^2 + y^2 = 9$
4. $\left(\sqrt{3}; 4\right)$ and $\left(\sqrt{3}; -4\right)$
5. a) $A\left(6\sqrt{10}; 2\sqrt{10}\right)$ or $A\left(-6\sqrt{10}; -2\sqrt{10}\right)$
   b) $A\left(4\sqrt{5}; 8\sqrt{5}\right)$ or $A\left(-4\sqrt{5}; -8\sqrt{5}\right)$
6. a) $x^2 + y^2 = 13$
   b)
   c)
   d) $PQ = 2\sqrt{13}$ units.
   e) $y = -\frac{3}{2}x$
   f) $y = \frac{2}{3}x + \frac{13}{3}$

Exercise 7 – 4: Equation of a circle with centre at $(a; b)$

1. a) Yes
   b) Yes
   c) No, coefficients of x^2 term and y^2 term are different.
   d) No, cannot be written in general form $(x-a)^2 + (y-b)^2 = r^2$
   e) Yes
   f) No, r^2 must be greater than zero.
2. a) $x^2 + (y-4)^2 = 9$
   b) $x^2 + y^2 = 25$
   c) $(x+2)^2 + (y-3)^2 = 40$
   d) $(x-p)^2 + (y+q)^2 = 6$
   e) $\left(x+\frac{1}{2}\right)^2 + \left(y-\frac{3}{2}\right)^2 = 10$
   f) $(x-1)^2 + (y+5)^2 = 26$
3. a) Centre: $(0; 2)$, $r = 5$ units
   b) Centre: $\left(-\frac{1}{2}; 0\right)$, $r = 2$ units
   c) Centre: $(2; -1)$, $r = \sqrt{10}$ units
   d) Centre: $(-1; 3)$, $r = 5$ units
   e) Centre: $(-3; -4)$, $r = \sqrt{30}$ units
   f) Centre: $\left(\frac{1}{3}; 2\right)$, $r = \sqrt{8}$ units
   g) Centre: $(-4; 5)$, $r = \sqrt{20}$ units
   h) Centre: $(3; -3)$, $r = \sqrt{12}$ units
4. $x^2 + (y+4)^2 = 20$ or $x^2 + (y-4)^2 = 20$
5. $(x+2)^2 + (y+2)^2 = 25$
6. a) $(x-4)^2 + (y-4)^2 = 13$
   b) $M\left(\frac{7}{2}; \frac{13}{2}\right)$
   c) $m_{MN} \times m_{KL} = -5 \times \frac{1}{5} = -1$
   d) $P(2; 1)$
   e) $y = \frac{3}{2}x - 2$
7. $(x-1)^2 + (y+3)^2 = 37$
8. a) $x^2 + y^2 = 34$
   b) $(x-2)^2 + (y+3)^2 = 34$
   c)

   d)

Centre of the shifted circle: $(2; 3)$

9.

Exercise 7 – 5: Equation of a tangent to a circle

1. a) $m = -\frac{2}{3}$
   b) $m_{\perp} = \frac{3}{2}$
2. a) $m = \frac{5}{3}$
   b) $m_{\text{tangent}} = -\frac{3}{5}$
3. $y = 7x + 19$
4. $y = -\frac{3}{4}x - \frac{1}{2}$
5. $y = 2x - 8$
6. a) $(x+4)^2 + (y-8)^2 = 136$
   b) 18
   c) $y = \frac{3}{5}x + 24$
7. a) $P(-4; -2)$ and $Q(2; 4)$
   b) $PQ = 6\sqrt{2}$
   c) $M(-1; 1)$
   d) $m_{PQ} \times m_{OM} = -1$
   e) $y = -2x - 10$ and $y = -\frac{1}{2}x + 5$
   f) $S(-10; 10)$
   h) $y = -2x + 10$ and $y = -2x - 10$

Exercise 7 – 6: End of chapter exercises

1. a) $(x)^2 + (y-5)^2 = 25$
   b) $(x-2)^2 + y^2 = 16$
   c) $(x+5)^2 + (y-7)^2 = 324$
   d) $(x+2)^2 + y^2 = 9$
   e) $(x+5)^2 + (y+3)^2 = 3$
2. a) $(x-2)^2 + (y-1)^2 = 4$
   b) $(0;1)$ and $(2;3)$
3. a) $(x+3)^2 + (y+2)^2 = 20$
   b) $(x-3)^2 + (y-1)^2 = 17$
4. a) $(-9;6)$, $r = 6$ units
   b) $(2;9)$, $r = \sqrt{2}$ unit
   c) $(-5;-7)$, $r = \sqrt{12}$ units
   d) $(0;-4)$, $r = \sqrt{23}$ units
   e) $(2;-3)$, $r = 2$ units
5. a) $(0;16)$, $(0;-4)$, $(-8;0)$, $(8;0)$
   b) $(0;0)$, $(-8;0)$
6. a) $(-3;6)$ and $r = 5$ units
   b) $(-2;4)$ and $r = \sqrt{20}$ units
   c) $(0;-4)$ and $r = \sqrt{23}$ units
   d) $(3;0)$ and $r = 5$ units
   e) $(\frac{5}{2};-\frac{3}{2})$ and $r = \frac{\sqrt{31}}{2}$ units
   f) $(3n;-5n)$ and $r = \sqrt{43}n$ units
7. a) $m = \frac{1}{12}$
   b) $m_{\perp} = -12$
8. a) $(0;4)$, $(2;4)$
   b) $m = \frac{1}{3}$
9. $y = -\frac{1}{5}x + \frac{14}{5}$
10. $y = -\frac{7}{9}x - \frac{22}{3}$
11. $y = \frac{1}{2}x - 4$
12. a) $(x+4)^2 + (y-2)^2 = 61$
    b) $p = 3$
    c) $y = -\frac{6}{5}x - 15$
13. a) $y = -\frac{1}{4}x + \frac{17}{4}$
    b) $y = -\frac{3}{4}x + \frac{25}{4}$
    c) $y = -\frac{4}{3}x + 9$
    d) $y = -\frac{3}{2}x + \frac{21}{2}$
14. $y = x - 10$ and $y = x + 10$
15. a) $y = -x + 8$
    b) $B(5;3)$
    c) $(x-4)^2 + (y-4)^2 = 2$
    d) The circle must be shifted 4 units down and 4 units to the left.
    e) $(x-6)^2 + (y-5)^2 = 18$

8 Euclidean geometry

Exercise 8 – 1: Revision

1. $z = 30°$
2. a) $OD = 3$ cm
   b) $AD = 8$ cm
   c) $AB = 4\sqrt{5}$cm
3. a) $R\hat{Q}S$, $Q\hat{S}O$
   b) $P\hat{O}S = 2a$
4. a) $O\hat{D}C = 35°$
   b) $C\hat{O}D = 110°$
   c) $C\hat{B}D = 55°$
   d) $B\hat{A}D = 90°$
   e) $A\hat{D}B = 45°$
6. $m = \frac{1}{4}n$
7. $y = 17$ mm
9. a) $A\hat{B}T = 90° - \frac{x}{2}$
   b) $O\hat{B}A = \frac{x}{2}$
   c) $\hat{C} = 90° - \frac{x}{2}$

Exercise 8 – 2: Ratio and proportion

1. a) $p = 5$
   b) $p = 7$
   c) $p = \frac{1}{2}$
   d) $p = 2$
2. Red $= 60$ sweets, blue $= 40$ sweets and yellow $= 60$ sweets,
3. Substance $B = 35{,}71$ kg
5. a) $(a+b) : b$
   b) $\frac{a}{2a} = \frac{1}{2}$
   c) $\frac{2a+b}{3a+b}$
   d) $2a : b$
6. $\frac{\text{area } ABF}{\text{area } ABCD} = \frac{9}{25}$
7. $AC = 20$ m, $CB = 20$ m
8. $TQ = 30$ m, $PT = 15$ m
9. a) $1{,}2$ m^2
   b) Perimeter: 414,8 cm; Area 11 700 cm^2
   c) 108×108cm^2

Exercise 8 – 3: Proportionality of polygons

1. a) $MN = 17{,}15$ cm
   b) Area $\triangle OPQ = 44{,}12$ cm^2
2. $t = 12$ cm, $p = 18$ cm, $q = 30$ cm
3. $\sqrt{5}x^2$
4. a) $FH = 19$ mm
   b) 76 mm^2

Exercise 8 – 4: Proportionality of triangles

1. a) $\frac{BC}{AB} = \frac{3}{2}$
   b) $\frac{AB}{AC} = \frac{2}{5}$
   c) 30 mm
2. $QP = 16$ cm and $PR = 10$ cm
5. a) "The line joining **the mid-points of two sides** of a triangle is **parallel** to the third side and equal to **half the length of the third side.**"
   c) Converse: a line through the mid-point of one side of a triangle, parallel to a second side, bisects the third side.

Exercise 8 – 5: Proportion theorem

1. a) $\frac{1}{4}$
   b) $\frac{9}{16}$
3. 4,87

Exercise 8 – 6: Similar polygons

1. a) Similar, all angles $= 90°$ and sides are in the same proportion.
   b) Not enough information is given. Corresponding angles are equal.
   c) Not similar, sides are not in the same proportion.
2. a) True
   b) False
   c) False
   d) True
   e) False
   f) True
   g) False
   h) True

Exercise 8 – 7: Similarity of triangles

1. b) 1,6 cm
2. c) 21,6 mm
3. $AE = 2$ cm, $EC = 12{,}3$ cm and $BE = 2{,}7$ cm

Exercise 8 – 8: Similarity of triangles

1. Yes
2. $ED = 4$ cm
4. c) $FE = 20$ cm

Exercise 8 – 9: Theorem of Pythagoras

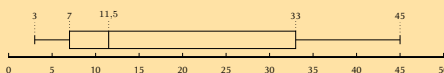
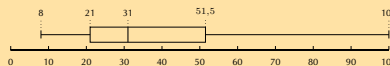
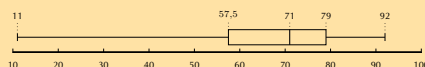
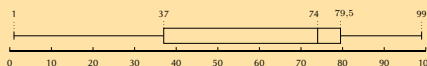
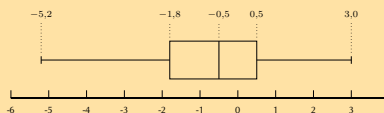
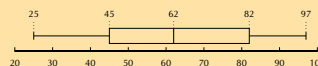
1. $6\sqrt{3}$cm
2. $a = 1$ unit and $b = 2\sqrt{3}$ units
3. b) 3 units

Exercise 8 – 10: End of chapter exercises

1. $SV = 17{,}5$ units
2. $\frac{DS}{SZ} = \frac{5}{3}$
3. $IJ = 25{,}7$ m and $KJ = 22{,}4$ m
4. $FH = 23{,}5$ cm
5. $\frac{HJ}{KI} = \frac{40}{9}$
6. $BC = 8{,}3$ m, $CF = 16{,}7$ m, $CD = 8{,}7$ m, $CE = 5{,}6$ m, $EF = 11{,}1$ m
7. $XZ = 49$ cm and $YZ = 46{,}96$ cm
9. c) $2\sqrt{3}$ cm
12. a) $> W\hat{R}S = 90°$
    b) $\hat{W} = 40°$
    c) $\hat{P}_1 = 40°$
14. a) $\hat{A} = \hat{D}_4 = \hat{E}_2 = \hat{D}_2 = x$

9 Statistics

Exercise 9 – 1: Revision

1. a) skewed right
   b) symmetric
   c) skewed left
   d) skewed right
   e) skewed left
   f) symmetric
2. a) mean = $18{,}5$; five number summary (3; 7; $11{,}5$; 33; 45); skewed right.

   b) mean = $40{,}83$; five number summary (8; 21; 31; $51{,}5$; 100); skewed right

   c) mean = $65{,}4$; five number summary (11; $57{,}5$; 71; 79; 92); skewed left

   d) mean = $40{,}83$; five number summary (1; 37; 74; $79{,}5$; 99); skewed left

   e) mean = $-0{,}6$; five number summary ($-5{,}2$; $-1{,}8$; $-0{,}5$; $0{,}5$; 3); close to symmetric/slightly skewed left

   f) mean = $61{,}83$; five number summary (25; 45; 62; 82; 97) ; symmetric

3. a) $\bar{x} = 5{,}9;\ \sigma^2 = 12{,}54;\ \sigma = \pm 3{,}54;\ 63\%$
   b) $\bar{x} = 5{,}5;\ \sigma^2 = 7{,}65;\ \sigma = \pm 2{,}77;\ 70\%$
   c) $\bar{x} = 43;\ \sigma^2 = 735{,}17;\ \sigma = \pm 27{,}11;\ 67\%$

4. a)
- Mean = 5
- $\sigma^2 = 9$
- $\sigma = \pm 3$

b)
- Mean = 8,58
- $\sigma^2 = 30,91$
- $\sigma = \pm 5,56$

c)
- Mean = 3,88
- $\sigma^2 = 1,47$
- $\sigma = \pm 1,21$

d)
- Mean = $-1,66$
- $\sigma^2 = 11,47$
- $\sigma = \pm 3,39$

e)
- Mean = 54,07
- $\sigma^2 = 1406,07$
- $\sigma = \pm 37,50$

5. a) A: 3,84. B: 3,82.
b) A: $\pm 0,121$. B: $\pm 0,184$.
c) Supermarket A

6. a) 47,87
b) $\pm 0,27$
c) 3

7. 13 and 20

8. 1,72 m

9. a) 47,5%
b) 29

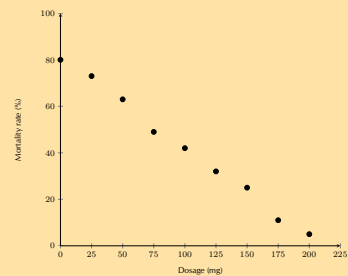
Exercise 9 – 2: Intuitive curve fitting

1. a) quadratic
b) exponential
c) linear
d) linear
e) exponential
f) quadratic

2. a) strong, positive linear function
b) $y = 0,6x + 1$ - learner answer may vary
c) 16 - learner answer may vary
d) 40 - learner answer may vary

3. a) Exponential
b) 33 554 432

4. a) Quadratic
b) temperature $= 18,33$, yield $= 6,17$

5. a)

b) strong, negative linear
c) $y = 0,38x + 80$ - learner answer may vary
d) 210,53 mg
e) Extrapolation - trend different outside of data range

Exercise 9 – 3: Least squares regression analysis

1. a) $\hat{y} = 0,62 + 0,57x$
b) $\hat{y} = -6,57 + 0,45x$
c) $\hat{y} = -22,61 + 1,70x$

2. a) $\hat{y} = 9,07 + 1,26x$
b) $\hat{y} = -29,09 + -5,84x$
c) $\hat{y} = 9,45 + 1,33x$
d) $\hat{y} = -12,44 + -3,71x$
e) $\hat{y} = -1,94 + 3,25x$
f) $\hat{y} = -5,64 + 0,72x$
g) $\hat{y} = 3,52 + -0,13x$
h) $\hat{y} = 14,55 + -0,03x$
i) $\hat{y} = 16,94 + 0,20x$
j) $\hat{y} = 5,14 + -7,03x$

3. a) $\hat{y} = 0,19 + 5,70x$
b) $\hat{y} = 18,62 - 0,09x$
c) $\hat{y} = 3,61 + 2,33x$

4. c) $\hat{y} = 858,48 + 164,06x$
d) R 3319,42
e) 13

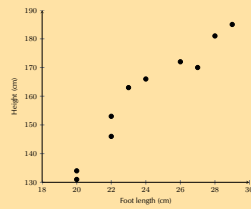
5. a)

English % (x)	Mathematics % (y)	xy	x^2
28	35	980	784
33	36	1188	1089
30	34	1020	900
45	45	2025	2025
45	50	2250	2025
55	40	2200	3025
65	50	3250	4225
70	65	4550	4900
76	85	6460	5776
65	70	4550	4225
85	80	6800	7225
90	90	8100	8100
$\sum = 742$	$\sum = 740$	$\sum = 46\,673$	$\sum = 47\,324$

b) $\hat{y} = 6{,}01 + 0{,}89x$

c) $50{,}51\%$

d) $77{,}52\%$

6. a)

b) Strong (or fairly strong), positive, linear trend

c) $\hat{y} = 27{,}56 + 5{,}50x$

e) $146{,}36$ cm

f) $29{,}53$ cm

Exercise 9 – 4: Correlation coefficient

1. a) $r = -0{,}29$, negative, weak
   b) $r = 0{,}68$, positive, moderate
   c) $r = 0{,}91$, positive, very strong
   c) $r = 0{,}00$, no correlation
   d) $r = 0{,}14$, positive, very weak.
   e) $r = 0{,}83$, positive, strong.

2. a) $r = -0{,}95$, negative, very strong.
   b) $r = -0{,}40$, negative, weak.

3. a) $-0{,}48$, negative, weak.
   b) $0{,}50$, positive, moderate.

4. a)

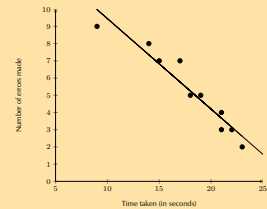
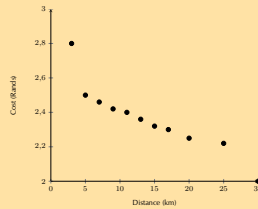
City	Degrees N (x)	Average temp. (y)	xy	x^2	$(x - \bar{x})^2$	$(y - \bar{y})^2$
Cairo	43	22	946	1849	15,21	0,64
Berlin	53	19	1007	2809	193,21	14,44
London	40	18	720	1600	0,81	23,04
Lagos	6	32	192	36	1095,61	84,64
Jerusalem	31	23	713	961	65,61	0,04
Madrid	40	28	1120	1600	0,81	27,04
Brussels	51	18	918	2601	141,61	23,04
Istanbul	39	23	897	1521	0,01	0,04
Boston	43	23	989	1849	15,21	0,04
Montreal	45	22	990	2025	34,81	0,64
Total:	391	228	8492	16 851	1562,9	173,6

b) $\hat{y} = 33{,}38 - 0{,}27x$

d) $-0{,}81$

e) strong, negative, linear

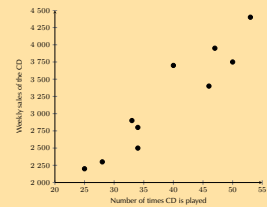
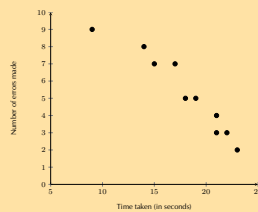
f) $31{,}04$

5. a)

b) $\hat{y} = 2{,}67 - 0{,}02x$

c) $r = -0{,}92$

d) very strong, negative, linear

e) 46 kilometres

6. a)

b) When more time is taken to complete the task, the learners make fewer errors.

c)

d) $r = -0{,}96$

e) 8

f) strong, negative

7. a)

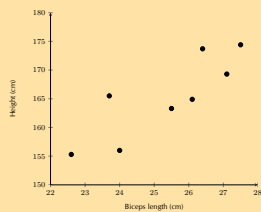
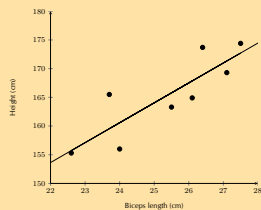
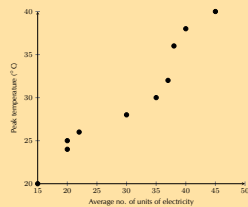
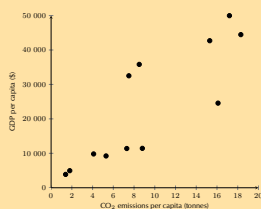
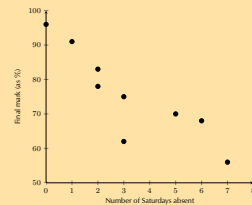
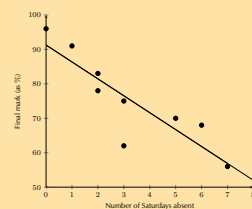
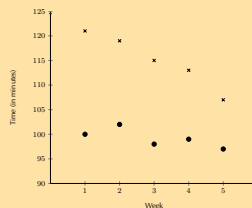
b) $\hat{y} = 293{,}06 + 74{,}28x$

c) $r = 0{,}95$

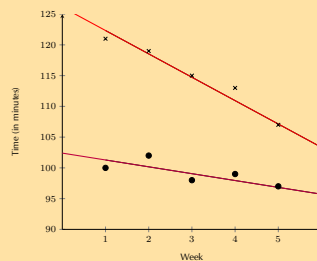
d) 3650 (to the nearest 50)

e) very strong, positive

Exercise 9 – 5: End of chapter exercises

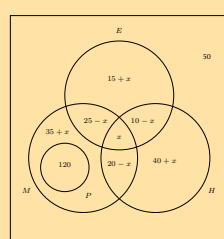
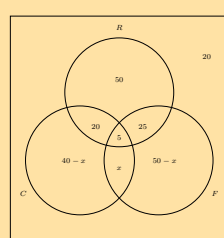
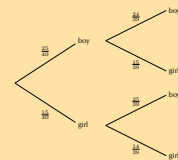
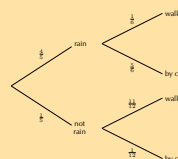
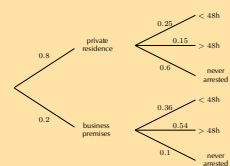
1. a) 16%
   b) $81{,}5\%$
2. $4{,}69$ machine hours
3. a) $\hat{y} = 86\,893{,}33 + 3497{,}14x$
   b) R 111 373,31, R 114 870,45.
   c) 13 months
4. a) $\hat{y} = 601{,}28 + 3{,}59x$
   b) r$=0{,}86$. Strong, positive, linear.
   c) 528 burgers
   d) R 2360,38
5. a)

   b) $\hat{y} = 77{,}32 + 3{,}47x$
   c)

   d) $r = 0{,}85$
   e) strong, positive, linear
6. a) Yes, $r = 0{,}98$
   b) 11
7. a)

   b) $\hat{y} = 11{,}36 + 0{,}61x$
   c) $0{,}96$
   d) Very strong, positive, linear
   e) 55 units, extrapolation
   f) quadratic
   g) temperature $= 16{,}54$, units $= 9{,}44$
8. a)

   b) $y = 2500x + 1000$ - learner answer may vary
   c) $\hat{y} = 1134{,}00 + 2393{,}74x$
   d) $r = 0{,}85$
   e) strong, positive, linear
   f) $0{,}24$ tonnes
9. a)

   b)

   c) $r = -0{,}87$
   d) The greater the number of Saturdays absent, the lower the mark.
   e) 72%
10. a)

   b) Both data sets show negative, linear trends. The trend in Grant's data appears to be more rapidly decreasing than the trend in Christie's data.
   c)

   d)
   e) No
   f) ninth week

10 Probability

Exercise 10 – 1: The product and addition rules

1. a) Dependent
   b) Independent
   c) Independent
   d) Independent
   e) Dependent
2. a) No
   b) Yes
3. a) Dependent
   b) Independent
4. a) No
   b) Yes
5. a) $\frac{1}{36}$
   b) $\frac{1}{4}$
   c) $\frac{25}{36}$
   d) $\frac{5}{18}$
   e) $\frac{11}{36}$
6. a) $\frac{6}{35}$
   b) $\frac{12}{35}$
   c) $\frac{17}{35}$
   d) Injuries, suspensions etc.
7. a) $\frac{1}{3}$
   b) $\frac{2}{3}$
   c) $\frac{2}{9}$
8. a) $\frac{2}{3}$
   b) $\frac{4}{15}$
   c) $\frac{1}{3}$
   d) Yes
9. a) No
   b) Yes
10. a) 0,5
    b) $\frac{5}{7}$
11. a) $\frac{1}{3}$
    b) $\frac{1}{36}$
    c) Yes
12. a) Learner dependent.
    b) 0,44
    c) 0,56
13. a) $\frac{4}{15}$
    b) $\frac{2}{5}$
    c) $\frac{7}{9}$
    d) $\frac{2}{9}$
    e) $\frac{8}{45}$
14. 55%
15. a) $\frac{2}{3}$
    b) $\frac{1}{3}$
    c) $\frac{1}{6}$
    d) $\frac{2}{3}$
    e) Independent

Exercise 10 – 2: Venn and tree diagrams

1. a) 115
   b) 16
   c) $\frac{73}{115}$
   d) $\frac{68}{115}$
   e) $\frac{1}{23}$
   f) $\frac{20}{23}$
   g) $\frac{81}{115}$
   h) $\frac{39}{115}$
   i) $\frac{24}{115}$
2. a)
   b) 5
   c) $\frac{5}{32}$
   d) $\frac{33}{64}$
3. a)
   b) 10
   c) $\frac{1}{10}$
   d) $\frac{17}{20}$
   e) No
4. a)

   b) $\frac{25}{52}$
   c) $\frac{45}{52}$
   d) dependent
5. a)

   b) $\frac{2}{15}$
   c) $\frac{41}{60}$
6. a)

   b) 0,48
   c) 0,5
   d) 0,875
   e) i) 4; ii) 7

Exercise 10 – 3: Contingency tables

1. a) Variables: gender and no. of accidents. Purpose: to determine if gender is related to the number of accidents a driver is involved in.

   b)

	≤ 2 accidents	> 2 accidents	Total
Female	210	90	300
Male	140	60	200
Total	350	150	500

   c) Yes

2. a) $\frac{3}{4}$

   b) $\frac{1}{8}$

   c) 81

   d)

	Malaria	No malaria	Total
Male	27	189	216
Female	81	567	648
Total	108	756	864

3. a) 120

   b) 280

   c) 84

   d)

	Reaction time $< 1{,}5$ s	Reaction time $> 1{,}5$ s	Total
≤ 40 years	84	36	120
> 40 years	196	84	280
Total	280	120	400

4. a)

	Flu	No flu	Total
Placebo	228	60	288
Treatment	12	252	264
Total	240	312	552

   b) $\frac{13}{23}$

   c) $\frac{11}{23}$

   d) $\frac{21}{46}$

   e) Dependent

   f) $\frac{21}{22}$

   g) $\frac{5}{24}$

   h) Yes

   i) No

5. a)

	Sick	Healthy	Total
Positive	999	9	1008
Negative	1	8991	8992
Total	1000	9000	10 000

   b) 0,01

   c) 0,0001

Exercise 10 – 4: Number of possible outcomes if repetition is allowed

1. 60
2. $1{,}0995 \times 10^{12}$
3. 100 000
4. 8 000 000
5. 2880
6. a) 175 760 000

   b) 60 761 421

   c) 3 114 752
7. 314

Exercise 10 – 5: Factorial notation

1. a) 6
   b) 720
   c) 12
   d) 40 320
   e) 120
   f) 738
   g) 359
   h) $\frac{1}{15}$
   i) -28
   j) 216
   k) 72
2. a) 239 500 800
   b) $1{,}49 \times 10^{-12}$
   c) 574 560
   d) 960
   e) 20 736

Exercise 10 – 6: Number of choices in a row

1. 720
2. 120
3. 120
4. 612
5. 27 907 200
6. 60
7. a) $1{,}31 \times 10^{12}$
   b) 32 760
8. a) 720
   b) 24
   c) 144
9. a) 120
   b) 216
10. a) 120
    b) 48
    c) 24
11. a) 39 916 800
    b) 13 824
    c) 725 760
    d) 967 680

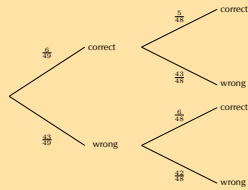
Exercise 10 – 7: Number of arrangements of sets containing alike objects

1. a) 362 880
   b) 30 240
   c) 3360
   d) 6720
   e) 3360
2. a) 3 628 800
   b) 75 600
   c) 11 760
   d) 22 680
   e) 360
3. a) 5040
   b) 823 543
   c) 16 807
   d) 210
4. a) 5040
   b) 35
5. a) 8
   b) 56
   c) 70
6. a) 256
   b) 24
   c) 128

Exercise 10 – 8: Solving probability problems using the fundamental counting principle

1. a) 5040
   b) 120
   c) 0,114
2. a) 72
   b) 0,5
3. a) 6 227 020 800
   b) 9 676 800
   c) 0,513
4. a) 40 320
   b) 0,25
5. $\frac{1}{12}$
6. $\frac{1}{210}$
7. a) $\frac{1}{240}$
   b) $\frac{1}{8}$
   c) 0,271
8. a) 9 239 076 000
   b) 0,025

Exercise 10 – 9: End of chapter exercises

1. a) 10 000
   b) $\frac{1}{10}$
   c) $\frac{1}{10}$
   d) $\left(\frac{1}{10}\right)^4 = 0{,}0001$
   e) 0,9997
2. a) $1{,}01 \times 10^{10}$
   b)

   c) $\frac{6}{49}$
   d) $\frac{5}{48}$
   e) $\frac{6}{48}$
   f) $\frac{6}{49}$
   g) $7{,}15 \times 10^{-8}$
3. 11,14%
4. 0,35
5. a) 1 000 000 000
   b) 10 000 000
   c) $\frac{1}{2}$
   d) 0,002
6. a) 3 326 400
   b) $\frac{1}{165}$
7. 38,89%
8. $\frac{2}{35}$
9. a) 0,18
   b) 0,9
   c) 0,2
   d) 0,2
10. a) 5040
    b) $\frac{1}{35}$
11. $\frac{1}{38}$
12. 12
13. a)

	Sick	Healthy	Total
Positive	99	999	1098
Negative	1	98 901	98 902
Total	100	99 900	100 000

    b) 0,09